Short-long First passage percolation on two dimensional torus

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Joint work with Rick Durrett

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First passage percolation on $N \times N$ torus

- Suppose one agent is present at each vertex of a $N \times N$ torus.
- At time 0 the center receives an information.
- Each neighbor of the center gets the information independently after time $Exp(1/4)$.
- In general whenever a vertex is informed, each of its uninformed neighbor gets the information independently after $Exp(1/4)$.
- $\mathcal{B}_t$ is the set of vertices informed by time t. $\mathcal{B}_0 = \{(0,0)\}$.

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Theorem (Shape theorem: Cox and Durrett 1981)

1. *Diameter of $\mathcal{B}_t$ grows linearly. It has an asymptotic shape.*
2. *If T_N is the time when every agent is informed, then T_N/N converges to a number.*

Short-long FPP on $N \times N$ torus (Aldous '07)

- $\mathcal{B}_t$ is the set of vertices informed by time t. $\mathcal{B}_0 = \{(0,0)\}$.
- Information spreads from vertex i to j after time $Exp(\nu_{ij}$,

$$\nu_{ij} = \begin{cases} 1/4 & \text{if } j \text{ is a (nearest) neighbor of } i \\ \lambda_N/N^2 & \text{if not.} \end{cases}$$

- Question: How does $\mathcal{B}_t$ grow?

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Simplify: (1) Deterministic nearest neighbor growth, (2) formulate on the (real) torus $\Gamma(N)$. Let $\mathcal{C}_t$ be the subset informed by time t.

- $\mathcal{C}_t$ starts with one "center" chosen uniformly from $\Gamma(N)$ at time 0.
- Each center corresponds to a disk, whose radius grows as $r(s) = s/\sqrt{2\pi}$.
- At time t birth rate of new centers is $\lambda_N \, |\mathcal{C}_t|$.
- The location of each new center is chosen uniformly from the torus.
- If the new point lands at $x \in \mathcal{C}_t$, nothing happens

Main result

"$\lambda_N \ll N^{-3}$" is similar to nearest neighbor case. Let $\lambda_N = N^{-\alpha}$ with $\alpha < 3$.

Theorem

There is a positive random variable M with $E(M) = 1$ such that for

$$\psi(t) := (2 - 2\alpha/3)N^{\alpha/3} \log N - N^{\alpha/3} \log M + tN^{\alpha/3},$$

$$\lim_{N\to\infty} P\left(\sup_{s\in(-\infty,t]} \left|N^{-2}\left|\mathcal{C}_{\psi(s)}\right| - h(s)\right| > \delta\right) = 0$$

for any $\delta > 0$, where $h(t)$ is a solution of

$$h(t) = 1 - \exp\left(-\int_{-\infty}^{t} \frac{(t-s)^2}{2} h(s)\, ds\right).$$

Main result (continued)

If for $\epsilon > 0$ $h_\epsilon(t)$ satisfies

$$h_\epsilon(t) = 1 - \exp\left(-\epsilon \int_{-\infty}^{\log(3\epsilon)} \frac{(t-s)^2}{2} e^s \, ds - \int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} h_\epsilon(s) \, ds\right)$$

for $t \geq \log(3\epsilon)$, and $h_\epsilon(\log(3\epsilon)) = \epsilon$, then $h_\epsilon(t) \downarrow h(t)$ uniformly on compact sets as $\epsilon \downarrow 0$.

Remarks:

- $|\mathcal{C}_t|$ reaches ϵN^2 by time $O(N^{\alpha/3} \log N)$,
- $|\mathcal{C}_t|$ increases from ϵN^2 to ρN^2 by time $O(N^{\alpha/3})$.

Branching balloon process

Begin by studying $\mathcal{A}_t$.

- A_t is the sum of the areas of all of the disks at time t,
- new centers are born at rate $N^{-\alpha}A_t$ at uniformly chosen location.

Theorem (LLN for $\mathcal{A}_t$)

$EA_t \approx a(t) = (1/3)N^{2\alpha/3}\exp\left(N^{-\alpha/3}t\right)$.
$A_t/a(t) = M_t + U_t$, M_t *is a positive* L^2 *martingale,* $U_t \to 0$ *a.s. and in* L^2.
So $A_t/a(t) \to M$ *a.s. and in* L^2.

Comparison between $\mathcal{C}_t$ and $\mathcal{A}_t$

$\mathcal{C}_t$ and $\mathcal{A}_t$ can be coupled so that $\mathcal{C}_t \subseteq \mathcal{A}_t$
Let $\sigma(\epsilon) = \inf\{t : A_t \geq \epsilon N^2\}$ and $\tau(\epsilon) = \inf\{t : |\mathcal{C}_t| \geq \epsilon N^2\}$.

Theorem

If $S(\epsilon) = N^{\alpha/3}[(2 - 2\alpha/3)\log N + \log(3\epsilon)]$ *so that* $a(S(\epsilon)) = \epsilon N^2$, *then*

$$N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) \to -\log M \ \ \textit{in probability}$$

Clearly $|\mathcal{C}_t| \leq A_t$. *In the other direction*

$$E\,|\mathcal{C}_t| \geq EA_t - c(a(t))^2/N^2.$$

Using this for any $\gamma > 0$

$$\limsup_{N\to\infty} P[\tau(\epsilon) > \sigma((1+\gamma)\epsilon)] \leq P(M < (1+\gamma)\epsilon^{1/3}) + 2\epsilon^{1/3}/\gamma.$$

So $\tau(\epsilon) \sim N^{\alpha/3}[(2 - 2\alpha/3)\log N - \log M] + \log(3\epsilon)N^{\alpha/3} = \psi(\log(3\epsilon))$.

Asymptotic behavior of $\mathcal{C}_t$ (Heuristic)

- $A_{\psi(t)}/N^2 \to e^t$ in probability.
- Number the generations of centers in $\mathcal{C}_t$ starting with those at time $\psi(\log(3\epsilon))$ as generation 0.
- Using $A_{\psi(t)} \approx \left|\mathcal{C}_{\psi(t)}\right|$ for $t \leq \log(3\epsilon)$, if $h_k(t)$ is the fraction of area covered by centers of generations $j \in \{0, 1, \ldots, k\}$, then

$$1 - h_0(t) \approx \exp\left(-\int_{-\infty}^{\log(3\epsilon)} \frac{(t-s)^2}{2} e^s \, ds\right).$$

- $$1 - f_{k+1}(t) \approx (1 - f_k(t)) \exp\left(-\int_{\log(3\epsilon)}^{t} \frac{(t-s)^2}{2} (f_k(s) - f_{k-1}(s)) \, ds\right).$$

- $h_k(t) \uparrow h_\epsilon(t)$ as $k \uparrow \infty$, and $h_\epsilon(t) \downarrow h(t)$ as $\epsilon \downarrow 0$.

Open question:

What about $\tau(1)$?

Future plan: To consider the case where the information percolation rate ν_{ij} depends on the distance between i and j.

Thank You