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Lieb-Robinson Bounds and Existence of the Thermodynamic Limit for a Class of Irreversible Quantum Dynamics

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Dedicated to Robert A. Minlos at the occasion of his 80th birthday

Abstract. We prove Lieb-Robinson bounds and the existence of the thermodynamic limit for a general class of irreversible dynamics for quantum lattice systems with time-dependent generators that satisfy a suitable decay condition in space.

1. Introduction

For a quantum many-body Hamiltonian describing bulk matter we expect that the Heisenberg dynamics converges in the thermodynamic limit to a well-defined one-parameter flow of transformations on the observable algebra. Early results of this kind were obtained for quantum spin systems [**29, 25, 27**], which were followed by generalizations that included examples of irreversible dynamics described by a semigroup of completely positive unit preserving maps [**9, 28**] . See, e.g., [**6, 8, 3, 30, 16**] for results on the thermodynamic limit of a number of examples of semigroups of completely positive maps. In this work we study a general class of irreversible dynamics for quantum lattice systems with generators that are sums of bounded terms that may depend on time and that satisfy a suitable decay condition in space.

Following the argument of [**26**] propagation bounds of Lieb-Robinson type [**14**] have recently been used to prove a number of new results on the existence of the thermodynamic limit [**17, 1, 19, 2**]. These recent developments were made possible by recent extensions and improvements of the Lieb-Robinson bounds themselves [**10, 20, 12, 18, 21**].

Lieb-Robinson type bounds for irreversible dynamics were, to our knowledge, first considered in [**11**] in the classical context and in [**24**] for a class of quantum lattice systems with finite-range interactions. Here, we will extend those results by

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proving a Lieb-Robinson bound for lattice models with a dynamics generated by both Hamiltonian and dissipative interactions with suitably fast decay in space and that may depend on time. See Assumption 1 for the precise conditions. Then, we use our result to prove the existence of the thermodynamic limit of the dynamics in the sense of a strongly continuous one-parameter flow of completely positive unit preserving maps.

Our results are applicable to a wide range of applications in statistical mechanics, quantum optics, and quantum information and computation. In each of those areas, it is often necessary to incorporate dissipative and time-dependent terms in the generator of the dynamics. Fortunately, there is a large number of interesting systems defined on a lattice, which so far is the only setting accessible by our methods to prove Lieb-Robinson bounds. It is probably not a coincidence that proofs of the existence of the thermodynamic limit of the dynamics have so far also been mostly restricted to lattice systems. Here, 'lattice' has to be interpreted loosely to mean a discrete set of points that are typically thought of as distributed in space. In the case to the positions of atoms in a crystal, these positions can indeed be described by a lattice, but all one needs is the structure of a metric graph satisfying some regularity conditions. The detailed setup is given in Section 2.

The existence of the thermodynamic limit is important as a fundamental property of any model meant to describe properties of bulk matter. In particular, such properties should be essentially independent of the size of the system which, of course, in any experimental setup will be finite. In the past five years, Lieb-Robinson bounds have been to prove a variety of interesting results about condensed matter systems. See [**22**] for a brief overview of the applications of Lieb-Robinson bounds.

The paper is organized as follows. First, we describe the general setup necessary to state our main results, which we do in Section 2. In this section we also state the three main theorems we prove in this paper. Theorem 1 states that solution of the differential equation (master equation) defined by finite volume generators we consider is a well-defined quantum dynamics, i.e., a continuous family of completely positive unit preserving maps on the algebra of observables. The proof of this theorem is obtained by standard methods, but for completeness we included it here in Section 3. Theorem 2 is the Lieb-Robinson bound, i.e., the propagation estimate for irreversible dynamics. Again the theorem is stated in Section 2 and then proved in Section 4. Theorem 3, the existence of the thermodynamic limit, is proved in Section 5.

2. Setup and main results

We consider quantum systems consisting of components associated with the vertices $x \in \Gamma$, where Γ is a countable set equipped with a metric d. We assume that there exists a function $F : [0, \infty) \to (0, \infty)$ such that:
i) F is uniformly integrable over Γ, i.e.,

$$\|F\| := \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x, y)) < \infty,$$

and
ii) F satisfies

$$C := \sup_{x,y\in\Gamma} \sum_{z\in\Gamma} \frac{F(d(x,z))F(d(y,z))}{F(d(x,y))} < \infty.$$

Having such a set Γ and a function F that satisfies i) and ii), for any $\mu > 0$ the function

$$F_\mu(x) = e^{-\mu x} F(x),$$

also satisfies i) and ii) with $\|F_\mu\| \le \|F\|$ and $C_\mu \le C$.

The Hilbert space of states of the subsystem at $x \in \Gamma$ is denoted by $\mathcal{H}_x$. For any finite subset $\Lambda \subset \Gamma$ the Hilbert associated with Λ is

$$\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x.$$

The algebra of observables supported in Λ is defined by

$$\mathcal{A}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{B}(\mathcal{H}_x),$$

where $\mathcal{B}(\mathcal{H}_x)$ is the set of bounded linear operators on $\mathcal{H}_x$. If $\Lambda_1 \subset \Lambda_2$, then we may identify $\mathcal{A}_{\Lambda_1}$ in a natural way with the subalgebra $\mathcal{A}_{\Lambda_1} \otimes \mathbb{1}_{\Lambda_2\setminus\Lambda_1}$ of $\mathcal{A}_{\Lambda_2}$, and simply write $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$. The algebra of local observables is then defined as

$$\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\subset\Gamma} \mathcal{A}_\Lambda.$$

The C^*-algebra of quasi-local observables $\mathcal{A}_\Gamma$ is the norm completion of $\mathcal{A}_\Gamma^{\mathrm{loc}}$. See [**4, 5**] for more details about this mathematical framework.

The support of the observable $A \in \mathcal{A}_\Lambda$ is the minimal set $X \subset \Lambda$ for which $A = A' \otimes \mathbb{1}_{\Lambda\setminus X}$ for some $A' \in \mathcal{A}_X$.

The generator of the dynamics is defined for each finite volume $\Lambda \subset \Gamma$, and, in general, contains both Hamiltonian interactions and dissipative terms, which we allow to be time-dependent. The Hamiltonian terms are described by an interaction $\Phi(t,\cdot)$ which, for all $t \in \mathbb{R}$, is a map from a set of subsets of Γ to $\mathcal{A}_\Gamma$, such that for each finite set $X \subset \Gamma$, $\Phi(t,X) \in \mathcal{A}_X$ and $\Phi(t,X)^* = \Phi(t,X)$. The dissipative part is described by terms of Lindblad form determined, for each finite $X \subset \Gamma$, a set of operators $L_a(t,X) \in \mathcal{A}_X$, $a = 1,\ldots,N(X)$. We can allow for the case $N(X) = \infty$, if we impose a suitable convergence condition on the resulting series for the generator. Then, for any finite set $\Lambda \subset \Gamma$ and time $t \in \mathbb{R}$ we define the family of bounded linear maps $\mathcal{L}_\Lambda : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, i.e. $\mathcal{L}_\Lambda \in \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$, as follows: for all $A \in \mathcal{A}_\Lambda$,

$$\Psi_Z(t)(A) = i[\Phi(t,Z), A] + \sum_{a=1}^{N(Z)} [L_a^*(t,Z) A L_a(t,Z) - \frac{1}{2}\{L_a(t,Z)^* L_a(t,Z), A\}] \tag{2.1}$$

$$\mathcal{L}_\Lambda(t)(A) = \sum_{Z\subset\Lambda} \Psi_Z(t)(A), \tag{2.2}$$

where $\{A,B\} = AB + BA$, is the anticommutator of A and B. The operators $\Psi_Z(t)$ can be regarded as bounded linear transformations on $\mathcal{A}_X$, for any $X \subset \Lambda$

that contains Z, which are then of the form $\Psi_Z(t) \otimes \mathrm{id}_{\mathcal{A}_{X\setminus Z}}$. The norm of these maps, in general, depends on X, but they are uniformly bounded as follows:

$$\|\Psi_Z(t)\| \leq 2\|\Phi(t,Z)\| + 2\sum_{a=1}^{N(Z)} \|L_a(t,Z)\|^2$$

If $N(z) = \infty$, we can insure uniform boundedness by assuming that the sums $\sum_{a=1}^{\infty} \|L_a(t,Z)\|^2$ converge. It is more general and more natural, however, to assume that the maps $\Psi_Z(t)$, defined on $\mathcal{A}_Z$ are completely bounded. By definition, $\Psi \in \mathcal{B}(\mathcal{A}_Z)$ is called *completely bounded* if for all $n \geq 1$, the linear maps $\Psi \otimes \mathrm{id}_{M_n}$, defined on $\mathcal{A}_X \otimes M_n$, where $M_n = \mathcal{B}(\mathbb{C}^n)$ are the $n \times n$ complex matrices, are bounded with uniformly bounded norm. This means, we can define the cb-norm of Ψ by

$$\|\Psi\|_{\mathrm{cb}} = \sup_{n\geq 1} \|\Psi \otimes \mathrm{id}_{M_n}\| < \infty$$

See [**7, 23**] for more information on completely bounded maps. Assuming that $\|\Psi_Z(t)\|_{\mathrm{cb}}$ is finite is more general than assuming that the series $\sum_{a=1}^{\infty} \|L_a(t,Z)\|^2$ converges which, however, is a useful sufficient condition for it. In particular, there are situations where the sum in (2.1) only converges in the strong operator topology but nevertheless yields a well-defined limit with finite cb-norm.

Assumption 1. *Given (Γ, d) and F as described at the beginning of this section, the following hypotheses hold:*

(1) *For all finite $\Lambda \subset \Gamma$, $\mathcal{L}_\Lambda(t)$ is continuous in t, and hence uniformly continuous on compact intervals.*
(2) *There exists $\mu > 0$ such that for every $t \in \mathbb{R}$*

$$\|\Psi\|_{t,\mu} := \sup_{s\in[0,t]} \sup_{x,y\in\Lambda} \sum_{Z\ni x,y} \frac{\|\Psi_Z(s)\|_{\mathrm{cb}}}{F_\mu(d(x,y))} < \infty. \tag{2.3}$$

where $\|\cdot\|_{\mathrm{cb}}$ denotes the cb-norm of completely bounded maps [**23**].

Note that

$$\|\mathcal{L}_\Lambda(t)\| \leq \sum_{Z\subset\Lambda} \|\Psi_Z(t)\| \leq \sum_{x,y\in\Lambda} \sum_{Z\ni x,y} \|\Psi_Z(t)\|_{\mathrm{cb}} \leq \|\Psi\|_{t,\mu}|\Lambda|\|F\|.$$

We define

$$M_t = \|\Psi\|_{t,\mu}|\Lambda|\|F\|\,. \tag{2.4}$$

Then by (2.3) one gets $M_s \leq M_t$ for $s < t$.

Fix $T > 0$ and, for all $A \in \mathcal{A}_\Lambda$, let $A(t), t \in [0,T]$ be a solution of the initial value problem

$$\frac{d}{dt}A(t) = \mathcal{L}_\Lambda(t)A(t), \quad A(0) = A. \tag{2.5}$$

Since $\|\mathcal{L}_\Lambda(t)\| \leq M_T < \infty$, this solution exists and is unique by the standard existence and uniqueness results for ordinary differential equations. For $0 \leq s \leq t \leq T$, define the family of maps $\{\gamma_{t,s}^\Lambda\}_{0\leq s\leq t} \subset \mathcal{B}(\mathcal{A}_\Lambda, \mathcal{A}_\Lambda)$ by $\gamma_{t,s}^\Lambda(A) = A(t)$, where $A(t)$ is the unique solution of (2.5) for $t \in [s,T]$ with initial condition $A(s) = A$. Then, the *cocycle property*, $\gamma_{t,s}(A(s)) = A(t)$, follows from the uniqueness of the solution of (2.5). Recall that a linear map $\gamma : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras is called *completely positive* if the maps $\gamma \otimes \mathrm{id} : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n$

are positive for all $n \geq 1$. Here M_n stands for the $n \times n$ matrices with complex entries, and positive means that positive elements (i.e., elements of the form A^*A) are mapped into positive elements. See, e.g., [**23**] for a discussion of the basic properties of completely positive maps. In particular, we shall use the property that every unit preserving (i.e. $\gamma(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$) completely positive map γ, is a contraction: $\|\gamma(A)\| \leq \|A\|$.

As a preliminary result we prove the following Theorem 1 in Section 3. It extends the well-known result for time-independent generators of Lindblad form [**15**] to the time-dependent case.

THEOREM 1. *Let $\mathcal{A}$ be a C^*-algebra, $T > 0$, and for $t \in [0, T]$, let $\mathcal{L}(t)$ be a norm-continuous family of bounded linear operator on $\mathcal{A}$. If*
(i) $\mathcal{L}(t)(\mathbb{1}) = 0$;
(ii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*) = \mathcal{L}(t)(A)^*$;
(iii) for all $A \in \mathcal{A}$, $\mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) \geq 0$;
then the maps $\gamma_{t,s}$, $0 \leq s \leq t \leq T$, defined by equation (2.5), are a norm-continuous cocycle of unit preserving completely positive maps.

It is straightforward to check that the $\mathcal{L}_\Lambda(t)$ defined in (2.2) satisfy properties (i) and (ii). Property (iii), which is called *complete dissipativity*, follows immediately from the observation

$$\mathcal{L}(t)(A^*A) - \mathcal{L}(t)(A^*)A - A^*\mathcal{L}(t)(A) = \sum_{Z \subset \Lambda} \sum_{a=1}^{N(Z)} [A, L_a(t,Z)]^*[A, L_a(t,Z)] \geq 0\,.$$

Therefore, using this result, we conclude that, under Assumption 1, for all finite $\Lambda \subset \Gamma$, the maps $\gamma^{\Lambda}_{t,s}$, $0 \leq s \leq t$, form a norm-continuous cocycle of completely positive and unit preserving maps.

Section 4 is devoted to proving a Lieb-Robinson bound for the irreversible dynamics $\gamma^{\Lambda}_{t,s}$. For reversible dynamics given by the one-parameter group of automorphisms τ_t describing the Heisenberg dynamics generated by a Hamiltonian, Lieb-Robinson bounds take the following form: there are constants $v, \mu > 0$ such that for $A \in A_X$ and $B \in \mathcal{A}_Y$,

$$\|[A, \tau_t(B)]\| \leq C(A,B)e^{-\mu(d(X,Y) - v|t|)}\,, \tag{2.6}$$

where $d(X,Y)$ denotes the distance between X and Y and $C(A,B)$ is a prefactor, which typically has the form $c\|A\|\,\|B\|\min(|X|,|Y|)$, for a suitable norm $\|\cdot\|$ on the observables A and B, and a suitable measure $|\cdot|$ on the size of the supports X and Y. Bounds of this form are sufficient to determine the approximate support of the time-evolved observable $\tau_t(B)$. See, e.g., [**2**, Lemma 3.1].

For irreversible dynamics, it turns out to be both natural and convenient to consider a slightly more general formulation. For $X \subset \Lambda$, let $\mathcal{B}_X$ denote the subspace of $\mathcal{B}(\mathcal{A}_X)$ consisting of all completely bounded linear maps that vanish on $\mathbb{1}$. See the discussion directly preceding Assumption 1 for the definition of complete boundedness and the cb-norm $\|\cdot\|_{\mathrm{cb}}$. It is important for us that all operators of the form

$$\mathcal{K}_X(B) = [A, B] + \sum_{a=1}^{N} [L_a^* B L_a - \frac{1}{2}\{L_a^* L_a, B\}]\,,$$

where $A, L_a \in \mathcal{A}_X$, belong to $\mathcal{B}_X$, with

$$\|\mathcal{K}_X\|_{\rm cb} \leq 2\|A\| + 2\sum_{a=1}^{N} \|L_a\|^2.$$

In particular, operators of the form $[A, \cdot]$ appearing in the standard Lieb-Robinson bound (2.6) are a special case of this general form. Then, we can regard $\mathcal{K}_X$ as a linear transformation on $\mathcal{A}_Z$, for all Z such that $X \subset Z$, by tensoring it with $\mathrm{id}_{\mathcal{A}_{Z\setminus X}}$, and all these maps will be bounded with norm less then $\|\mathcal{K}_X\|_{\rm cb}$.

THEOREM 2. *Suppose Assumption 1 holds. Then the maps $\gamma^{\Lambda}_{t,s}$ satisfy the following bound. For $X, Y \subset \Lambda$, and any operators $\mathcal{K} \in \mathcal{B}_X$ and $B \in \mathcal{A}_Y$ we have that*

$$\|\mathcal{K}(\gamma^{\Lambda}_{t,s}(B))\| \leq \frac{\|\mathcal{K}\|_{\rm cb}\,\|B\|}{C_\mu} e^{\|\Psi\|_{t,\mu}C_\mu|t-s|} \sum_{x\in X}\sum_{y\in Y} F_\mu(d(x,y))\,.$$

Note that the bound in this theorem is *uniform* in Λ. This is important for the proof of existence of the thermodynamic limit of the dynamics, which is one of the main applications of the present paper.

The setup for the analysis of the thermodynamic limit can be formulated as follows. Let Γ be an infinite set such as, e.g., the hypercubic lattice $\mathbb{Z}^\nu$. We prove the existence of the thermodynamic limit for an increasing and absorbing sequence of finite subsets $\Lambda_n \subset \Gamma$, $n \geq 1$, by showing that for each $A \in \mathcal{A}_X$, $(\gamma^{\Lambda_n}_{t,s}(A))_{n\geq 1}$ is a Cauchy sequence in the norm of $\mathcal{A}_\Gamma$. To this end we have to suppose that Assumption 1 (2) holds *uniformly* for all Λ_n, i.e., we can replace Λ in (2.3) by Γ.

THEOREM 3. *Suppose that Assumption 1 holds and, in addition, that (2.3) holds for $\Lambda = \Gamma$. Then, there exists a strongly continuous cocycle of unit-preserving completely positive maps $\gamma^{\Gamma}_{t,s}$ on $\mathcal{A}_\Gamma$ such that for all $0 \leq s \leq t$, and any increasing exhausting sequence of finite subsets $\Lambda_n \subset \Gamma$, we have*

$$\lim_{n\to\infty} \|\gamma^{\Lambda_n}_{t,s}(A) - \gamma^{\Gamma}_{t,s}(A)\| = 0, \tag{2.7}$$

for all $A \in \mathcal{A}_\Gamma$.

3. Finite volume dynamics

Let $\mathcal{L}(t)$, $t \geq 0$, denote a family of operators on a C^*-algebra $\mathcal{A}$ satisfying the assumptions of Theorem 1 and for $0 \leq s \leq t$ consider the maps $\mathcal{A} \ni A \mapsto \gamma_{t,s}(A)$ defined by the solutions of (2.5) with initial condition A at $t = s$. Without loss of generality we can assume $s = 0$ in the proof of the theorem because, if we denote $\tilde{\mathcal{L}}(t) = \mathcal{L}(t+s)$, then $\gamma_{t,s} = \tilde{\gamma}_{t-s,0}$, where $\tilde{\gamma}_{t,0}$ is the maps determined by the generators $\tilde{\mathcal{L}}(t)$.

The maps $\gamma_{t,s}$ satisfy the equation

$$\gamma_{t,s} = \mathrm{id} + \int_s^t \mathcal{L}(\tau)\gamma_{\tau,s}d\tau. \tag{3.1}$$

In our proof of the complete positivity of $\gamma_{t,0}$ we will an expression for $\gamma_{t,0}$ as the limit of an Euler product, i.e.. approximations $T_n(t)$ defined by

$$T_n(t) = \prod_{k=n}^{1} \left(\mathrm{id} + \frac{t}{n}\mathcal{L}(\frac{kt}{n})\right)\,. \tag{3.2}$$

The product is taken in the order so that the factor with $k = 1$ is on the right.

Lemma 1. *Let $\mathcal{L}(t)$, $t \geq 0$, denote a family of operators on a C^*-algebra $\mathcal{A}$ satisfying the assumptions of Theorem 1. Then, uniformly for all $t \in [0,T]$,*

$$\lim_{n\to\infty} \|T_n(t) - \gamma_{t,0}\| = 0\,,$$

where $T_n(t)$ is defined by (3.2).

Proof. From the cocycle property established in Section 2, we have

$$\gamma_{t,0} = \prod_{k=n}^{1} \gamma_{t\frac{k}{n},t\frac{k-1}{n}}.$$

Now, consider the difference

$$\begin{aligned} T_n(t) - \gamma_{t,0} &= \prod_{k=n}^{1} \left(\mathrm{id} + \tfrac{t}{n}\mathcal{L}(\tfrac{kt}{n})\right) - \prod_{k=n}^{1} \gamma_{t\frac{k}{n},t\frac{k-1}{n}} \\ &= \sum_{j=1}^{n} \left[\prod_{k=n}^{j+1} \left(\mathrm{id} + \tfrac{t}{n}\mathcal{L}(t\tfrac{k-1}{n})\right)\right] \left[\left(\mathrm{id} + \frac{t}{n}\mathcal{L}(t\tfrac{j-1}{n})\right) - \gamma_{t\frac{j}{n},t\frac{j-1}{n}}\right] \gamma_{t\frac{j-1}{n},0}. \end{aligned}$$

To estimate the norm of this difference we look at each factor separately.

Using the boundedness of $\mathcal{L}(t)$ and the fact that M_t, defined in (2.4), is increasing in t, the norm of the first factor is bounded from above by

$$\|\prod_{k=n}^{j+1} (\mathrm{id} + \frac{t}{n}\mathcal{L}(t\frac{k-1}{n}))\| \leq \prod_{k=n}^{1} (1 + \frac{t}{n}\|\mathcal{L}(t\frac{k-1}{n})\|) \leq (1 + \frac{t}{n}M_t)^n.$$

To bound the second factor notice that from (3.1) we obtain

$$\|\gamma_{t,s}\| \leq 1 + \int_s^t \|\mathcal{L}(\tau)\|\|\gamma_{t,s}\|d\tau.$$

Then by Gronwall inequality [**13**, Theorem 2.25] we have the following bound for the norm of the $\gamma_{t,s}$:

$$\|\gamma_{t,s}\| \leq e^{\int_s^t \|\mathcal{L}(\tau)\|d\tau} \leq e^{M_t(t-s)}.$$

Using again (3.1) we can rewrite the second factor as follows:

$$\begin{aligned} &\left(\mathrm{id} + \frac{t}{n}\mathcal{L}(t\frac{j-1}{n})\right) - \gamma_{t\frac{j}{n},t\frac{j-1}{n}} = \frac{t}{n}\mathcal{L}(t\frac{j-1}{n}) - (\gamma_{t\frac{j}{n},t\frac{j-1}{n}} - \mathrm{id}) \\ &= \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} \left(\mathcal{L}(t\frac{j-1}{n}) - \mathcal{L}(s)\gamma_{s,t\frac{j-1}{n}}\right) ds \\ &= \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} \left[\left(\mathcal{L}(t\frac{j-1}{n}) - \mathcal{L}(s)\right) - \mathcal{L}(s)(\gamma_{s,t\frac{j-1}{n}} - \mathrm{id})\right] ds \\ &= \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} \left(\mathcal{L}(t\frac{j-1}{n}) - \mathcal{L}(s)\right) ds - \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} \mathcal{L}(s) \int_{t\frac{j-1}{n}}^{s} \mathcal{L}(\tau)\gamma_{\tau,t\frac{j-1}{n}} d\tau ds. \end{aligned}$$

Therefore, the second factor is bounded from above by

$$\begin{aligned}\| \left(\mathrm{id} + \frac{t}{n}\mathcal{L}(t\frac{j-1}{n})\right) - \gamma_{t\frac{j}{n}, t\frac{j-1}{n}} \| &\le \frac{t}{n}\epsilon_n + M_t^2 \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} \int_{t\frac{j-1}{n}}^{s} e^{(\tau - t\frac{j-1}{n})M_t} d\tau ds \\ &\le \frac{t}{n}\epsilon_n + M_t^2 e^{\frac{t}{n}M_t} \int_{t\frac{j-1}{n}}^{t\frac{j}{n}} (s - t\frac{j-1}{n}) ds \\ &= \frac{t}{n}\left(\epsilon_n + M_t^2 e^{\frac{t}{n}M_t}\frac{t}{2n}\right),\end{aligned}$$

where $\epsilon_n \to 0$ as $t/n \to 0$ due to the uniform continuity of $\mathcal{L}(t)$ on the interval $[0, t]$.

The third factor can be estimated in a similar way:

$$\begin{aligned}\|\gamma_{t\frac{j-1}{n},0}\| &= \prod_{k=j-1}^{1} \|\gamma_{t\frac{k}{n}, t\frac{k-1}{n}}\| = \prod_{k=j-1}^{1} \|1 + \frac{t}{n}\mathcal{L}(s_k(\frac{t}{n}))\| \\ &\le \prod_{k=j-1}^{1}\left(1 + \frac{t}{n}\|\mathcal{L}(s_k(\frac{t}{n}))\|\right) \\ &\le (1 + \frac{t}{n}M_t)^n.\end{aligned}$$

Therefore, combining all these estimates we obtain

$$\begin{aligned}\|T_n(t) - \gamma_{t,0}\| &\le n(1 + \frac{t}{n}M_t)^n \frac{t}{n}\left(\epsilon_n + M_t^2 e^{\frac{t}{n}M_t}\frac{t}{2n}\right)(1 + \frac{t}{n}M_t)^n \\ &\le te^{2tM_t}\left(\epsilon_n + M_t^2 e^{\frac{t}{n}M_t}\frac{t}{2n}\right).\end{aligned}$$

This bound vanishes as $n \to \infty$. □

To prove Theorem 1 we use the Euler-type approximation established in Lemma 1. We show that the action of T_n on a positive operator gives a sequence of bounded from below operators such that the negative bounds vanish as n goes to ∞.

Proof of Theorem 1: First, we look at the each term in the Euler approximation T_n separately. For any t and s the complete dissipativity property (iii) of $\mathcal{L}(s)$, assumed in the statement of the theorem, implies

$$\begin{aligned}0 &\le (\mathrm{id} + t\mathcal{L}(s))(A^*)(\mathrm{id} + t\mathcal{L}(s))(A) = (A^* + t\mathcal{L}(s)(A^*))(A + t\mathcal{L}(s)(A)) \\ &= A^*A + tA^*\mathcal{L}(s)(A) + t\mathcal{L}(s)(A^*)A + t^2\mathcal{L}(s)(A^*)\mathcal{L}(s)(A) \\ &\le A^*A + t\mathcal{L}(s)(A^*A) + t^2\mathcal{L}(s)(A^*)\mathcal{L}(s)(A).\end{aligned}$$

Since $(\mathcal{L}(s)(A))^*(\mathcal{L}(s)(A)) \le \|\mathcal{L}(s)\|^2\|A\|$, one gets

$$0 \le (\mathrm{id} + t\mathcal{L}(s))(A^*A) + t^2\|\mathcal{L}(s)\|^2\|A\|^2 \tag{3.3}$$

$$\le (\mathrm{id} + t\mathcal{L}(s))(A^*A) + t^2M_s^2\|A\|^2. \tag{3.4}$$

Let us apply the above inequality to the operator$B$, where $B^*B := \|A\|^2 - A^*A$. Note that $\|B^*B\| \le \|A\|^2$, so $\|B\| \le \|A\|$.

$$0 \le (\mathrm{id} + t\mathcal{L}(s))(\|A\|^2 - A^*A) + t^2M_s^2\|A\|^2 \tag{3.5}$$

$$= \|A\|^2 - (\mathrm{id} + t\mathcal{L}(s))(A^*A) + t^2M_s^2\|A\|^2 \tag{3.6}$$

From the (3.3) and (3.5) we obtain

$$-t^2 M_s^2 \|A\|^2 \le (\mathrm{id} + t\mathcal{L}(s))(A^* A) \le (1 + t^2 M_s^2)\|A\|^2 \tag{3.7}$$

and therefore:

$$-(1 + t^2 M_s^2)\|A\|^2 \le (\mathrm{id} + t\mathcal{L}(s))(A^* A) \le (1 + t^2 M_s^2)\|A\|^2.$$

So we get

$$\|(\mathrm{id} + t\mathcal{L}(s))(A^* A)\| \le (1 + t^2 M_s^2)\|A\|^2 \ . \tag{3.8}$$

Now, in order to bound the approximation T_n we first derive the following auxiliary estimate. For any fixed $n \ge 1$ we have:

$$\prod_{k=n}^{1} (\mathrm{id} + s\mathcal{L}(ks))(A^* A) \ge -s^2 \|A\|^2 M_{ns}^2 (1 + \frac{1}{n-1})^{n-1} \sum_{k=0}^{n-1} D(s)^k, \tag{3.9}$$

where the value of s is chosen to be such that

$$D(s) := 1 + s^2 M_{ns}^2 < (1 + \frac{1}{n-1})^{n-1} / (1 + \frac{1}{n-2})^{n-2}, \tag{3.10}$$

with the convention that $(1 + \frac{1}{n-1})^{n-1} = 1$, for $n = 1$.

We prove this claim by induction. The statement holds for $n = 1$ by (3.5). Now, assume that (3.9) holds for $n - 1$. Then

$$\prod_{k=n-1}^{1} (\mathrm{id} + s\mathcal{L}(ks))(A^* A) + s^2 \|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \ge 0$$

Since the left-hand side is a positive operator, we can write it as $B^* B$. Then,

$$\begin{aligned}
\prod_{k=n}^{1} (\mathrm{id} + s\mathcal{L}(ks))(A^* A) &= (\mathrm{id} + s\mathcal{L}(ns))(B^* B) \\
&\quad - s^2 \|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\
&\ge -s^2 M_{ns}^2 \|B^* B\| - s^2 \|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \ .
\end{aligned}$$

Here, we used (3.5) and the fact that M_t is monotone increasing. This gives the following upper bound for $\|B^* B\|$:

$$\begin{aligned}
\|B^* B\| &\le \prod_{k=n-1}^{1} \|(\mathrm{id} + s\mathcal{L}(ks))(A^* A)\| + s^2 \|A\|^2 M_{(n-1)s}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\
&\le \prod_{k=n-1}^{1} (1 + s^2 M_{ks}^2)\|A\|^2 + s^2 \|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\
&\le \prod_{k=n-1}^{1} (1 + s^2 M_{ns}^2)\|A\|^2 + s^2 \|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k \\
&= \|A\|^2 D(s)^{n-1} + s^2 \|A\|^2 M_{ns}^2 (1 + \frac{1}{n-2})^{n-2} \sum_{k=0}^{n-2} D(s)^k
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\prod_{k=n-1}^{1}(1+s\mathcal{L}(ks))(A^*A)\\
&\geq -s^2M_{ns}^2\|A\|^2D(s)^{n-1}-s^2M_{ns}^2(s^2M_{ns}^2+1)(1+\frac{1}{n-2})^{n-2}\|A\|^2\sum_{k=0}^{n-2}D(s)^k\\
&\geq -s^2M_{ns}^2\|A\|^2(1+\frac{1}{n-1})^{n-1}D(s)^{n-1}-s^2M_{ns}^2(1+\frac{1}{n-1})^{n-1}\|A\|^2\sum_{k=0}^{n-2}D(s)^k\\
&\geq -s^2M_{ns}^2(1+\frac{1}{n-1})^{n-1}\|A\|^2\sum_{k=0}^{n-1}D(s)^k\ ,
\end{aligned}$$

where to pass to the second inequality we use our assumption on s (3.10). This completes the proof of the bound (3.9).

To finish the proof of the theorem we use Lemma 1 to approximate the propagator and put $s=\frac{t}{n}$ in the bound (3.9), which yields

$$\prod_{k=n}^{1}(1+\frac{t}{n}\mathcal{L}(\frac{kt}{n}))(A^*A)\geq -\frac{t^2}{n^2}\|A\|^2M_t^2(1+\frac{1}{n-1})^{n-1}\sum_{k=0}^{n-1}D(\frac{t}{n})^k. \tag{3.11}$$

Since $D(\frac{t}{n})^n=(1+\frac{t^2}{n^2}M_t^2)^n\to 1$ as $n\to\infty$, we get the estimate $D(\frac{t}{n})^k\leq 2$ for $1\leq k\leq n$. The right hand side of (3.11) is bounded from below by $-\frac{t^2}{n^2}\|A\|^2e\,M_t^2 2n$, which vanishes in the limit $n\to\infty$.

To show the complete positivity of $\gamma_{t,0}$ note that any generator $\mathcal{L}_\Lambda(t)$ satisfying the assumptions of the theorem can be considered as the generator for a dynamics on $\mathcal{A}\otimes\mathcal{B}(\mathbb{C}^n)$, for any $n\geq 1$, which satisfies the same properties, and which generates $\gamma_{t,s}\otimes\mathrm{id}$ acting on $\mathcal{A}\otimes\mathcal{B}(\mathbb{C}^n)$. By the arguments given above, these maps are positive for all n. Hence, the $\gamma_{t,s}$ are completely positive. □

4. Lieb-Robinson bound

Our derivation of the Lieb-Robinson bounds for $\gamma_{t,s}^\Lambda$ is based on a generalization of the strategy [**17**] for reversible dynamics, and on [**24**] for irreversible dynamics with time-independent generators. This allows us to cover the case of irreversible dynamics with time-dependent generators.

Proof of Theorem 2: Consider the function $f:[s,\infty)\to\mathcal{A}$ defined by

$$f(t)=\mathcal{K}\gamma_{t,s}^\Lambda(B),$$

where $\mathcal{K}\in\mathcal{B}_X$ and $B\in\mathcal{A}_Y$, as in the statement of the theorem. For $X\subset\Lambda$, let $X^c=\Lambda\setminus X$ and define $\mathcal{L}_{X^c}$ and $\bar{\mathcal{L}}_X$ by

$$\begin{aligned}
\mathcal{L}_{X^c}(t) &= \sum_{Z,Z\cap X=\emptyset}\mathcal{L}_Z(t)\\
\bar{\mathcal{L}}_X(t) &= \mathcal{L}_X(t)-\mathcal{L}_{X^c}(t).
\end{aligned}$$

Clearly, $[\mathcal{K}, \mathcal{L}_{X^c}(t)] = 0$. Using this property, we easily derive the following expression for the derivative of f:

$$\begin{aligned} f'(t) &= \mathcal{K}\mathcal{L}(t)\gamma^{\Lambda}_{t,s}(B) \\ &= \mathcal{L}_{X^c}(t)\mathcal{K}\gamma^{\Lambda}_{t,s}(B) + \mathcal{K}\bar{\mathcal{L}}_X(t)\gamma^{\Lambda}_{t,s}(B) \\ &= \mathcal{L}_{X^c}(t)f(t) + \mathcal{K}\bar{\mathcal{L}}_X(t)\gamma^{\Lambda}_{t,s}(B) \ , \end{aligned}$$

Let $\gamma^{X^c}_{t.s}$ be the cocycle generated by $\mathcal{L}_{X^c}(t)$. Then, using the expression for $f'(t)$ we find

$$f(t) = \gamma^{X^c}_{t,s} f(s) + \int_s^t \gamma^{X^c}_{t,r}\mathcal{K}\bar{\mathcal{L}}_X(r)\gamma^{\Lambda}_{r,s}(B)dr \ .$$

Since $\gamma^{X^c}_{t,s}$ is norm-contracting and $\|\mathcal{K}\|_{\rm cb}$ is an upper bound for the $\|\mathcal{K}\|$ regarded as an operator on $\mathcal{A}_\Lambda$, for all Λ, we obtain

$$\|f(t)\| \le \|f(s)\| + \|\mathcal{K}\|_{\rm cb}\int_s^t \|\bar{\mathcal{L}}_X(r)\gamma^{\Lambda}_{r,s}(B)\|dr. \tag{4.1}$$

Let us define the quantity

$$C_B(X,t) := \sup_{\mathcal{T}\in\mathbb{L}_X} \frac{\|\mathcal{T}\gamma^{\Lambda}_{t,s}(B)\|}{\|\mathcal{T}\|_{\rm cb}}.$$

Note that we use the norm $\|\mathcal{T}\|_{\rm cb}$, because, in contrast to the usual operator norm, it is independent of Λ. Then, we have the following obvious estimate:

$$C_B(X,s) \le \|B\|\delta_Y(X),$$

where $\delta_Y(X) = 0$ if $X\cap Y = \emptyset$ and $\delta_Y(X) = 1$ otherwise. From the definition of the space $\mathcal{B}_X$ we get that $\mathcal{T}(B) = 0$, when $\mathcal{T}\in\mathcal{B}_X$, since B has a support in Y and $Y\cap X = \emptyset$.
Therefore (4.1) implies that

$$C_B(X,t) \le C_B(X,s) + \sum_{Z\cap X\neq\emptyset}\int_s^t \|\mathcal{L}_Z(s)\|C_B(Z,s)ds.$$

Iterating this inequality we find the estimate:

$$C_B(X,t) \le \|B\|\sum_{n=0}^{\infty}\frac{(t-s)^n}{n!}\, a_n \ ,$$

where:

$$a_n \le \|\Psi\|^n_{t,\mu}C^{n-1}_{\mu}\sum_{x\in X}\sum_{y\in Y}F_\mu(d(x,y)),$$

for $n\ge 1$ and $a_0 = 1$, (recall that C_μ is a constant, that appears in a definition of F_μ). The following bound immediately follows from this estimate:

$$\|\mathcal{K}\gamma^{\Lambda}_{t,s}(B)\| \le \frac{\|\mathcal{K}\|_{\rm cb}\|B\|}{C_\mu}e^{\|\Psi\|_{t,\mu}C_\mu(t-s)}\sum_{x\in X}\sum_{y\in Y}F_\mu(d(x,y)).$$

Using definition of F_μ, we can rewrite this bound as

$$\|\mathcal{K}\gamma^{\Lambda}_{t,s}(B)\| \le \frac{\|\mathcal{K}\|_{\rm cb}\|B\|}{C_\mu}\|F\|\min(|X|,|Y|)e^{-\mu(d(X,Y)-\frac{\|\Psi\|_{t,\mu}C_\mu}{\mu}(t-s))}.$$

So the Lieb-Robinson velocity of the propagation for every $t \in \mathbf{R}$ is

$$v_{t,\mu} := \frac{\|\Psi\|_{t,\mu} C_\mu}{\mu}.$$

□

Note that the bound above depends only on the smallest of the supports of the two observables. Therefore, in a situation where it makes sense to consider the limit of infinite systems, one can get a non-trivial bound when one of the observables has finite support but the support of the other is of infinite size (e.g., say half the system).

We would also like to point out that with the argument given in [**21**], size of the support $|X|$, can be replaced by a suitable measure of the surface area of the support, which gives a better estimate for observables with large supports.

5. Existence of the thermodynamic limit

Our proof of existence of the thermodynamic limit mimics the method given in the paper [**17**].

Proof of Theorem 3: Denote $\mathcal{L}_n = \mathcal{L}_{\Lambda_n}$ and $\gamma_{t,s}^{\Lambda_n} = \gamma_{t,s}^{(n)}$. Let $n > m$, then $\Lambda_m \subset \Lambda_n$ since we have the exhausting sequence of subsets in Γ. We will prove that for every observable $A \in \mathcal{A}_X$ the sequence $(\gamma_{t,s}^n(A))$ is a Cauchy sequence. In order to do that for any local observable $A \in \mathcal{A}_X$ we consider the function

$$f(t) := \gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A) \ .$$

Calculating the derivative, we obtain

$$\begin{aligned} f'(t) &= \mathcal{L}_n \gamma_{t,s}^{(n)}(A) - \mathcal{L}_m \gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t)(\gamma_{t,s}^{(n)}(A) - \gamma_{t,s}^{(m)}(A)) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A) \\ &= \mathcal{L}_n(t) f(t) + (\mathcal{L}_n(t) - \mathcal{L}_m(t))\gamma_{t,s}^{(m)}(A). \end{aligned}$$

The solution to this differential equation is

$$f(t) = \int_s^t \gamma_{t,r}^{(n)}(\mathcal{L}_n(r) - \mathcal{L}_m(r))\gamma_{r,s}^{(m)} A dr.$$

Since $\gamma_{t,r}$ is norm-contracting, from this formula we get the estimate:

$$\begin{aligned} \|f(t)\| &\le \int_s^t \|(\mathcal{L}_n(r) - \mathcal{L}_m(r))\gamma_{r,s}^{(m)}(A)\| dr \\ &\le \sum_{z \in \Lambda_n \setminus \Lambda_m} \sum_{Z \ni z} \int_s^t \|\Psi_Z(r)(\gamma_{r,s}^{(m)}(A))\| dr. \end{aligned}$$

Using the Lieb-Robinson bound and the exponential decay condition (2.3), which we assumed holds uniformly in Λ, we find that

$$\begin{aligned}
\|f(t)\| &\leq \frac{\|A\|}{C_\mu}\int_s^t e^{\mu v_{r,\mu} r} \sum_{z\in\Lambda_n\setminus\Lambda_m}\sum_{Z\ni z}\|\Psi_Z(r)\|_{\mathrm{cb}}\sum_{x\in X}\sum_{y\in Z}F_\mu(d(x,y))dr \\
&\leq \frac{\|A\|}{C_\mu}\int_s^t e^{\mu v_{r,\mu} r} \sum_{z\in\Lambda_n\setminus\Lambda_m}\sum_{x\in X}\sum_{y\in\Gamma}\sum_{Z\ni z,y}\|\Psi_Z(r)\|_{\mathrm{cb}}F_\mu(d(x,y))dr \\
&\leq \frac{\|A\|}{C_\mu}\|\Psi\|_{t,\mu}\int_s^t e^{\mu v_{r,\mu} r}dr \sum_{z\in\Lambda_n\setminus\Lambda_m}\sum_{x\in X}\sum_{y\in\Gamma}F_\mu(d(x,y))F_\mu(d(y,z)) \\
&\leq \frac{\|A\|}{C_\mu}C_\mu\|\Psi\|_{t,\mu}\int_s^t e^{\mu v_{r,\mu} r}dr \sum_{z\in\Lambda_n\setminus\Lambda_m}\sum_{x\in X}F_\mu(d(x,z)) \\
&\leq \frac{\|A\|}{C_\mu}C_\mu\|\Psi\|_{t,\mu}\int_s^t e^{\mu v_{r,\mu} r}dr|X|\sup_{x\in X}\sum_{z\in\Lambda_n\setminus\Lambda_m}F_\mu(d(x,z)).
\end{aligned}$$

Since F_μ is exponentially decaying when the distance $d(x,z)$ is increasing, we note that for $n,m\to\infty$, the last sum is goes to zero. Thus

$$\|(\gamma_{t,s}^{(n)}-\gamma_{t,s}^{(m)})(A)\|\to 0,\ \ as\ n,m\to\infty.$$

Therefore the sequence $\{\gamma_{t,s}^{(n)}(A)\}_{n=0}^\infty$ is Cauchy and hence convergent. Denote the limit, and its extension to $\mathcal{A}_\Gamma$, as $\gamma_{t,s}^\Gamma$.

To show that $\gamma_{t,s}^\Gamma$ is strongly continuous we notice that for $0\leq s\leq t,r\leq T$, and any $A\in\mathcal{A}_\Gamma^{\mathrm{loc}}$, we have

$$\|\gamma_{t,s}^\Gamma(A)-\gamma_{r,s}^\Gamma(A)\|\leq\|\gamma_{t,s}^\Gamma(A)-\gamma_{t,s}^{(n)}(A)\|+\|\gamma_{t,s}^{(n)}(A)-\gamma_{r,s}^{(n)}(A)\|+\|\gamma_{r,s}^{(n)}(A)-\gamma_{r,s}^\Gamma(A)\|,$$

for any $n\in\mathbb{N}$ such that $A\in\mathcal{A}_{\Lambda_n}$. The strong continuity then follows from the strong convergence of $\gamma_{t,s}^{(n)}$ to $\gamma_{t.s}^\Gamma$, uniformly in $s\leq t\in[0,T]$, and the strong continuity of $\gamma_{t,s}^{(n)}$ in t. The continuity of the extension of $\gamma_{t,s}^\Gamma$ to all of $A\in\mathcal{A}_\Gamma$ follows by the standard density argument. The argument for continuity in the second variable, s, is similar. □

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References

[1] L. Amour, P. Levy-Bruhl, and J. Nourrigat. Dynamics and Lieb-Robinson estimates for lattices of interacting anharmonic oscillators. *Colloq. Math.* 118, no. 2, 609–648, 2010. arXiv:0904.2717.

[2] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, *Automorphic Equivalence within Gapped Phases of Quantum Lattice Systems*, arXiv:1102.0842.

[3] O. Bratteli and A. Kishimoto, *Generation of semigroups and two-dimensional quantum lattice systems*, J. Funct. Anal. **35** (1980) 344-368.
[4] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics*, 2 ed., vol. 1, Springer Verlag, 1987.
[5] ———, *Operator algebras and quantum statistical mechanics*, 2 ed., vol. 2, Springer Verlag, 1997.
[6] E.B. Davies, *Irreversible Dynamics of Infinite Fermion Systems*, Commun. Math. Phys. **55** (1977) 231–258.
[7] E.G. Effros and Z.-J. Ruan, *Operator Spaces*, Oxford University Press, 2000.
[8] M. Fannes and A. Verbeure, *Global thermodynamical stability and correlation inequalities*, J. Math. Phys. **19** (1978) 558–560.
[9] V. Gorini, A. Kossakowski, and E.C.G. Sudarshan, *Completely positive dynamical semigroups of N-Ievel systems*, J. Math. Phys. **17** (1976) 821–825
[10] M. B. Hastings, *Lieb-Schultz-Mattis in higher dimensions*, Phys. Rev. B **69** (2004) 104431.
[11] ———, *Locality in Quantum and Markov Dynamics on Lattices and Networks*, Phys. Rev. Lett. **93** (2004) 140402.
[12] M. B. Hastings and T. Koma, *Spectral gap and exponential decay of correlations*, Commun. Math. Phys. **265** (2006), 781–804, arxiv:math-ph/0507.4708.
[13] J.K. Hunter and B. Nachtergaele, *Applied Analysis*, World Scientific, Singapore, 2001.
[14] E.H. Lieb and D.W. Robinson, *The finite group velocity of quantum spin systems*, Commun. Math. Phys. **28** (1972), 251–257.
[15] G. Lindblad, *On the Generators o f Quantum Dynamical Semigroups* Commun. Math. Phys. **48** (1976) 119–130.
[16] T. Matsui, *Markov Semigroups on UHF Algebras*, Rev. Math. Phys., **5** (1993), 587–600.
[17] B. Nachtergaele, Y. Ogata, and R. Sims, *Propagation of correlations in quantum lattice systems*, J. Stat. Phys. **124** (2006), 1–13, arXiv:math-ph/0603064.
[18] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, *Lieb-Robinson bounds for harmonic and anharmonic lattice systems*, Commun. Math. Phys. **286** (2009), 1073–1098, arXiv:0712.3820.
[19] B. Nachtergaele, B. Schlein, R. Sims, S. Starr, and V. Zagrebnov, *On the existence of the dynamics for anharmonic quantum oscillator systems*, Rev. Math. Phys. **22** (2010), 207–231, arXiv:0909.2249.
[20] B. Nachtergaele and R. Sims, *Lieb-Robinson bounds and the exponential clustering theorem*, Commun. Math. Phys. **265** (2006), 119–130, arXiv:math-ph/0506030.
[21] ———, *Locality estimates for quantum spin systems*, New Trends in Mathematical Physics. Selected contributions of the XVth International Congress on Mathematical Physics (V. Sidovaricius, ed.), Springer Verlag, 2009, arXiv:0712.3318, pp. 591–614.
[22] ———, *Much Ado About Something: Why Lieb-Robinson bounds are useful*, IAMP News Bulletin, October 2010, pp 22-29, arXiv:1102.0835.
[23] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002.
[24] D. Poulin. Lieb-Robinson bound and locality for general Markovian quantum dynamics. *Phys. Rev. Lett.* **104**, 190401, 2010.
[25] D.W. Robinson, *Statistical Mechanics of Quantum Spin Systems. II*, Commun. Math. Phys. **7** (1968) 337–348.
[26] ———, *Properties of propagation of quantum spin systems*, Austr. Math. Soc. **19** (1976) 387–399.
[27] D. Ruelle, *Statistical Mechanics*, Benjamin, Reading, MA, 1969.
[28] H. Spohn, *Kinetic equations from Hamiltonian dynamics: Markovian limits*, Rev. Mod. Phys. **53** (1980) 569–615
[29] R.F. Streater, *On Certain Non-Relativistic Quantized Fields*, Commun. Math. Phys. **7** (1968) 93–98.
[30] T. Unnerstall, *The Dynamics of Infinite Open Quantum Systems*, Lett. Math. Phys. **20** (1990) 183–187.

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