MAX-TO-MEAN RATIO ESTIMATES FOR THE FUNDAMENTAL EIGENFUNCTION OF THE DIRICHLET LAPLACIAN

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ABSTRACT. We review upper and lower bound isoperimetric properties of the fundamental eigenfunction of the Dirichlet Laplacian and announce new reverse Hölder type inequalities for norms of this function in the case of wedge-like membrane.

CONTENTS

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1. Introduction

The following article serves two purposes. In the first instance, we review various isoperimetric mean-to-peak ratio inequalities and related inequalities estimating various norms of the fundamental mode of vibration of the fixed membrane problem. We also announce recent improvements in the case of a wedge-like membrane. Salient features that go into the proof of these inequalities, and some known extensions, will be highlighted.

To fix the notation, we let $\Omega \subset \mathbb{R}^d$, and consider the first eigenfunction of the Dirichlet Laplacian and its associated eigenvalue, which we will denote respectively by u and $\lambda > 0$. This problem is described by

$$(1.1) \qquad \begin{aligned} \Delta u + \lambda u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

It is well known that u can be taken to be positive. We are interested in sharp isoperimetric inequalities relating the various norms $\|u\|_\infty = \operatorname{ess\,sup} u$ (which we will denote sometimes as u_{max}), $\|u\|_p, \|u\|_q$ (for $q \geq p > 0$) to the eigenvalue λ and underlying geometric features such as volume $|\Omega|$ and surface area $|\partial\Omega|$. Here $\|u\|_m = \left(\int_\Omega u^m \, dx\right)^{1/m}$.

The oldest result we are aware of is attributed in the work of Titchmarsh (1958) to Minakshisundaram (1942) (see p. 190 of [42]) who proved that $\exists\, C$ such that

$$\|u\|_\infty \leq C\lambda^{(d-1)/4},$$

under the normalizing condition $\|u\|_2 = 1$. Minakshisundaram's proof uses an asymptotic technique reminiscent of the work of Titchmarsh which will not be discussed further.

Properly defined, mean-to-peak-ratio for the fundamental eigenfunction of the Dirichlet Laplacian was first defined by Payne and Stakgold [35, 41, 53] as

$$E = \frac{\int_\Omega u \, dx}{|\Omega|\, u_{max}}$$

in the context of shape optimization for a nuclear reactor operating at criticality. Indeed, for a homogeneous, monoenergetic, critical reactor, the neutron density, u, is the first eigenfunction of the Dirichlet Laplacian described in (1.1). The physical motivation offered [35] for studying its properties is that high mean-to-peak neutron density ratio guarantees adequate average power output without exceeding the maximum temperature due to metallurgical considerations. Shaefer and Sperb [47, 48] call this quantity "neutron density ratio". Sperb [50] simply calls it "efficiency ratio". We are interested in both upper and lower isoperimetric bounds for this ratio, and more generally for ratios of the form

$$\frac{\|u\|_q}{\|u\|_p}$$

where $q \geq p > 0$. This includes the case $q = \infty$. Rather than expressing known results in terms of E, we will simply write them in terms of u and λ.

Bounds for such ratios emulate two well-known classical isoperimetric inequalities, namely the Rayleigh-Faber-Krahn inequality [7, 40] and the Makai-Pólya inequality [28, 39] (see also [22, 46]). The first states that λ is minimized for the unit

ball in $\mathbb{R}^d$ (or unit disk when $d = 2$) of same volume (or area) as Ω,

$$(1.2) \qquad \lambda \geq \frac{C_d^{2/d}\, j_{d/2-1,1}^2}{|\Omega|^{2/d}}$$

where $j_{d/2-1,1}$ denotes the first positive zero of the Bessel function $J_{d/2-1}(x)$ and C_d the volume of the unit ball. The second, valid when Ω is convex, takes the form

$$(1.3) \qquad \lambda \leq \frac{\pi^2}{4}\, \frac{|\partial\Omega|^2}{|\Omega|^2}$$

with equality occurring, in the limit, for an infinite slab.

2. Payne-Rayner and Payne-Stakgold inequalities

For two dimensional domains, Payne and Rayner proved [32, 33] that for $\Omega \subset \mathbb{R}^2$

$$(2.1) \qquad \frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda}{4\pi}$$

with equality for the disk. This is a reverse Hölder inequality. It has the interpretation of being a couched $L^2 \geq 4\pi A$ classical isoperimetric inequality for the domain Ω with a conformal metric based on the eigenfunction u (see [11] for details). The proof of this theorem relies on Schwarz symmetrization (or decreasing rearrangement), the Faber-Krahn inequality, and the classical geometric isoperimetric inequality.

Attempting to generalize this result using a new monotonicity principle for an auxiliary problem and rearrangement techniques, Kohler-Jobin [25, 26] proved several extensions of this result to dimensions $d \geq 2$ from which she obtained the corollary

$$(2.2) \qquad \frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda^{d/2}}{2d\, C_d\, j_{d/2-1,1}^{d-2}}.$$

Among the many results obtained by Kohler-Jobin [25, 26] we mention the explicit isoperimetric inequalities (see (2.5) below)

$$(2.3) \qquad \frac{\|u\|_\infty}{\|u\|_1} \leq C_1(d)\lambda^{d/2}$$

and

$$(2.4) \qquad \frac{\|u\|_\infty}{\|u\|_2} \leq C_2(d)\lambda^{d/4}.$$

Again, equality holds when Ω is the d-ball. The constants $C_1(d), C_2(d)$ are explicit expressions in terms of Bessel functions and their corresponding zeros. Using Steiner symmetrization and an ingenious comparison result, Chiti [13, 14] (see also [3]) was able to circumvent the auxiliary problem developed by Kohler-Jobin and proved at once

$$(2.5) \qquad \frac{\|u\|_q}{\|u\|_p} \leq K(p,q,d)\,\lambda^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \text{ for } q \geq p > 0$$

and

$$(2.6) \qquad \frac{\|u\|_\infty}{\|u\|_p} \leq K(p,d)\,\lambda^{\frac{d}{2p}} \text{ for } p > 0.$$

Here

$$K(p,q,d) = (d\,C_d)^{\frac{1}{q}-\frac{1}{p}}\, j_{\frac{d}{2}-1,1}^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \frac{\left(\int_0^1 r^{d-1+q(1-\frac{d}{2})} J_{\frac{d}{2}-1}^q (j_{\frac{d}{2}-1,1}\, r)dr\right)^{\frac{1}{q}}}{\left(\int_0^1 r^{d-1+p(1-\frac{d}{2})} J_{\frac{d}{2}-1}^p (j_{\frac{d}{2}-1,1}\, r)dr\right)^{\frac{1}{p}}}.$$

Again, these results are isoperimetric, and equality holds when Ω is the unit ball in $\mathbb{R}^d$, and $K(p,d) = \lim_{q\to\infty} K(p,q,d)$.

Under specific conditions on the domain Ω one can improve some of these results or provide counterparts. For all convex domains $\Omega \subset \mathbb{R}^d$, Payne and Stakgold [35] proved

$$(2.7) \qquad\qquad \frac{\|u\|_\infty}{\|u\|_1} > \frac{\pi}{2\,|\Omega|}.$$

The equality sign holds in the limit for an infinite slab. The proof of the Payne and Stakgold ineq. uses the interior parallels method and does not actually require the convexity of Ω, but rather that the average curvature of $\partial\Omega$ is non-negative. A weaker version of (2.7), under the same restrictions on Ω states

$$(2.8) \qquad\qquad \frac{\|u\|_\infty}{\|u\|_1} > \frac{\sqrt{\lambda}}{|\partial\Omega|}.$$

These two statements are sharp in the sense that equality is assumed for an infinite slab. Note that (2.8) follows from (1.3) and (2.7). For a convex domain, Payne and Stakgold [41] also proved

$$(2.9) \qquad\qquad \frac{\|u\|_\infty}{\|u\|_2} \geq \sqrt{\frac{2}{|\Omega|}}$$

and, excising information about the underlying domain,

$$(2.10) \qquad\qquad \|u\|_2^2 \leq \frac{\pi}{4} \|u\|_\infty \|u\|_1.$$

The method of proof uses in an essential way gradient estimates, which follow from a strong version of the Hopf maximum principle, and rearrangement techniques. At the suggestion of van den Berg, Payne [30] conjectured that one ought to prove a bound of the form

$$(2.11) \qquad\qquad \|u\|_\infty \leq F(|\Omega|, \|u\|_2)$$

where F is a function that is independent of the eigenvalue or other geometric quantities. This would provide a counterpart to (2.9). For a planar domain, Hersch [21] suggested the conjecture

$$(2.12) \qquad\qquad \frac{\|u\|_2^2}{\|u\|_1^2} \geq \frac{\pi}{4} \frac{\sqrt{\lambda}}{|\partial\Omega|}$$

which would provide a counterpart to (2.1). Neither a disk, nor a rectangle, both of which were checked by Hersch, provide extrema for this inequality. Further information about this ratio for the fundamental mode of vibration can be inferred when the underlying domain Ω is star-shaped with respect to a point inside $\Omega \subset \mathbb{R}^d$. In this case, one can define the pure number of the domain, B, (see [21, 40] in terms

of the "Stützfunktion", $h(\xi) = \langle \xi, n \rangle$, where ξ denotes a point on the boundary $\partial\Omega$ and n the outward normal at ξ, by

$$B = \int_{\partial\Omega} \frac{1}{h(\xi)} \, d\xi.$$

Based on the Rellich identity [45],

$$\int_{\partial\Omega} h(\xi) \left(\frac{\partial u}{\partial n}\right)^2 d\xi = 2\lambda \int_\Omega u^2 dx,$$

Crooke and Sperb [15] (see also [21]) proved,

$$\tag{2.13} \frac{\|u\|_2^2}{\|u\|_1^2} \geq \frac{\lambda}{2B}.$$

Two-sided bounds in the spirit of the above have been been proved by several authors for generalizations of the above boundary value problem. Payne and Stakgold [41] considered the nonlinear problem

$$\tag{2.14} \begin{aligned} \Delta u + f(u) &= 0 \text{ in } \Omega \subset \mathbb{R}^d \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

for a given continuous function $f(u)$, with $f(0) = 0$. This includes problem (1.1) when $f(u) = \lambda u$. Existence and uniqueness of positive solutions is laid out in [53] where a more general Robin boundary condition is also treated. Assuming smoothness conditions on the boundary, strong Hopf maximum principles are developed for a functional of $u, |\nabla u|$ which lead to useful pointwise bounds for $|\nabla u|$ in terms of u and u_{max}. When coupled with standard rearrangement arguments one is then led to Payne-Stakgold type inequalities. The results are valid for domains with nonnegative average curvature at every point of the boundary, which is the case of convex domains. For this problem, the Payne-Rayner inequality, proved in [15, 34], takes the form

$$\tag{2.15} \left(\int_\Omega f(u) \, dx\right)^2 \geq 8\pi \int_\Omega F(u) \, dx$$

where $F(u) = \int_0^u f(t)dt$. Payne, Sperb and Stakgold [34] use a different functional in the Hopf maximum part of the argument, but otherwise similar techniques, to improve some of the inequalities cited above. This is also the case of the papers of Shaefer and Sperb [47, 48, 49], and Philippin [38]. One finally points out Sperb's work [51] where an auxiliary problem which echoes earlier work by Bandle [5] and Kohler-Jobin [25, 26] but "interpolates" differently between the torsional and membrane problems on Ω,

$$\tag{2.16} \begin{aligned} \Delta u + \lambda \left(u + \alpha \int_\Omega u \, dx\right) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where α is a fixed parameter in $(-\frac{1}{|\Omega|}, \infty)$, has been treated and similar inequalities for the mean-to-max ratio and different norms of the eigenfunctions were treated. For, example the Payne Rayner inequality reads

$$\tag{2.17} \|u\|_2^2 \leq \left(\frac{\lambda}{4\pi}(1 + \alpha|\Omega|)^2 - 2\alpha\right) \|u\|_1^2.$$

In addition to the original papers [32, 33], Payne-Rayner type inequalities are now part of the standard body of literature on Schwarz symmetrization (symmetric decreasing rearrangement). Classical works include [13, 14, 23, 24, 29, 55]. Recent work has focused on extensions to non-linear eigenvalue problems [1, 2], on problems with with a Gaussian weight [6], on a larger class of elliptic problems [12], on free membrane problems (with Neumann conditions) for a class of domains in $\mathbb{R}^d$ with controlled relative isoperimetric content [10], on eigenvalue problems for the p-Laplacian and pseudo-p-Laplacian [8, 9], on a Hessian eigenvalue problem [17], on deriving a bound for the pressure integral in a toroidal-plasma equilibrium [59], or for extracting geometric features for minimal surfaces as is the case in [54, 58].

Based on the results of Chiti [13, 14], van den Berg, [56] was able to prove an isoperimetric inequality relating $\|u\|_\infty$ to the inradius $\rho = \max\{\min_{y\in\partial\Omega} : |x-y| : x \in \Omega\}$ under the usual normalization condition $\|u\|_2 = 1$:

$$(2.18) \qquad \|u\|_\infty \leq C(d)\rho^{-d/2}.$$

Jazzing up the work to a spherical cap in $\mathbb{S}^{d-1}$ and sending the radius of this spherical cap to zero, he was able led to the following conjecture

$$(2.19) \qquad \|u\|_\infty \leq C(d)\rho^{-\frac{1}{6}-\frac{d}{2}} D^{-\frac{1}{6}}$$

where D denotes the diameter of Ω. This conjecture of van den Berg was motivated by a counterexample of Kröger to any assertion that [27]

$$(2.20) \qquad \|u\|_\infty \sim |\Omega|^{-1/2}.$$

Kröger's counterexample was a wedge $S_\alpha = \{(r,\theta) : 0 < r < (2\alpha/\pi)^{1/2}, 0 < \theta < \pi/\alpha\}$, $\alpha \geq 1$ for which $|S_\alpha| = 1$, but $\|u\|_\infty \to \infty$ as $\alpha \to \infty$. Results in this spirit appear in [57] but use probabilistic methods.

Pointwise bounds for the size of the eigenfunction have already been established by Payne and Stakgold [35] who showed, for Ω convex,

$$(2.21) \qquad u(P) \leq \sqrt{\lambda}\,\|u\|_\infty\, d(P,\partial\Omega)$$

and

$$(2.22) \qquad u(P) \leq \sqrt{\lambda}\,\|u\|_1\, \frac{d(P,\partial\Omega)}{|\Omega|}$$

where P designates the point at which u reaches its maximum. These results are related to work by Grieser and Jerison [20] who obtained estimates for both the location and size of the maximum value of u for a convex domain $\Omega \subset \mathbb{R}^2$. Their estimates compare the size and location of the ground state eigenfunction with the size and location of the ground state eigenfunction of an associated one dimensional Schrödinger operator. Grieser and Jerison have returned recently to this question for a class of domains that are very special perturbations of a rectangle [19]. We also note that Grieser [18], apparently unaware of Minakshisundaram's work, proved a general statement in the same spirit using the wave equation method for the case of an eigenfunction of the Laplacian on a compact manifold with boundary, with Dirichlet or Neumann boundary conditions, also assuming the normalization $\|u\|_2 = 1$.

3. INEQUALITIES FOR WEDGE-LIKE MEMBRANES

Throughout this section, we consider a 2-dimensional domain $\Omega \subset \mathcal{W}_\alpha = \{(r, \theta) : 0 \le \theta \le \pi/\alpha\}$, $\alpha \ge 1$. In 1955, Payne and Weinberger [36] proved the curious inequality

$$(3.1) \qquad \lambda \ge \lambda^* = \left(\frac{4\alpha(\alpha + 1)}{\pi} \int_\Omega h^2(r, \theta) r \, dr \, d\theta \right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2$$

where $h = r^\alpha \sin \alpha\theta$. Here (r, θ) are polar coordinates taken at the apex of the wedge, and $j_{\alpha,1}$ the first zero of the Bessel function $J_\alpha(x)$. Equality holds if and only if Ω is a circular sector $\mathcal{W}_\alpha$. This inequality improves on the Faber-Krahn inequality for certain domains (as is the case of certain triangles) and has the interpretation of being a version of Faber-Krahn in dimension $2\alpha + 2$ for solids of rotation [4, 31]. Note that α need not be an integer. The proof of this inequality relies on a geometric isoperimetric inequality for the quantity

$$v_0 = \int_\Omega h^2(r, \theta) r \, dr \, d\theta$$

which is optimized for the sector $\mathcal{W}_\alpha$, and a carefully crafted symmetrization argument. A counterpart to this inequality, involving a sectorial version of the pure number B, and isoperimetric in the sense that equality holds when $\Omega = \mathcal{W}_\alpha$, was proved by Sperb [52]. In more recent works, the Payne Weinberger inequality was extended to a wedge on the sphere $\mathbb{S}^2$ [43], and to wedges in $\mathbb{R}^d$ [44].

The natural question about a Payne-Rayner type inequality for wedge-like domain was first addressed by Philippin [37] in 1976 who proved

$$(3.2)$$
$$\left(\frac{2\alpha}{\pi} \right)^{\frac{1}{\alpha+1}} (2\alpha + 2)^{-\frac{2\alpha+1}{\alpha+1}} \left(\int_\Omega u \, h \, r dr d\theta \right)^2 \le v_0^{\frac{\alpha}{\alpha+1}} \left(\frac{\alpha + 2}{\alpha + 1} \frac{1}{\lambda^*} - \frac{1}{\lambda} \right) \int_\Omega u^2 \, r dr d\theta$$

Equality holds for the sector. This result is one of two possible extensions of Payne-Rayner suggested in in higher dimensions appearing in the original paper [33] "neither of which is satisfactory". The second of such "unsatisfactory" inequalities has been proved by Mossino [29] (see also Kesavan [23, 24]) who proved for a weighted isoperimetric version involving quasi-norms in Lorentz space. In [16] we showed that

$$(3.3) \qquad \left(\int_\Omega u^q \, h^{2-q} r dr d\theta \right)^{\frac{1}{q}} \le K(p, q, 2\alpha + 2) \lambda^{(\alpha+1)(\frac{1}{p} - \frac{1}{q})} \left(\int_\Omega u^p h^{2-p} r dr d\theta \right)^{\frac{1}{p}}$$

with $K(p, q, 2\alpha + 2)$ as given in the Chiti statement (2.5). The proof of (3.3) is an adaptation of the method of Chiti [13, 14]. As in Payne and Weinberger [36], we first consider the transformation $u = h \, w$. The function w satisfies

$$\begin{cases} \Delta w + \frac{1}{h} \langle \nabla h, \nabla w \rangle + \lambda w = 0, & \text{in } \Omega ; \\ \frac{\partial w}{\partial n} = 0, & \text{on } \Gamma_1; \\ w = 0, & \text{on } \Gamma_2. \end{cases}$$

where $\Gamma_1 = \partial\Omega \cap \{\theta = 0, \frac{\pi}{\alpha}\}$. One then performs rearrangement for functionals of the form

$$\int_\Omega w^p h^2 dx dy$$

and develop a Chiti comparison theorem for sectors to complete the proof.

Payne and Stakgold pointed out in [35] that for wedge-like membranes one ought to be able to improve (2.8). However, this task has not been implemented, to the best of our knowledge.

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