Inequalities for Schrödinger Operators and Applications to the Stability of Matter Problem

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Abstract. We review various inequalities for Schrödinger operators and show how they can be applied to solve the problem of stability of matter.

1. Uncertainty Principles in Quantum Mechanics

1.1. Introduction. One of the most important differences between quantum and classical mechanics is the **uncertainty principle**. Among many other things, it implies that position and momentum of a particle can not simultaneously take on definite values. To make this more quantitative, recall that the state of a quantum system is described by a **wave function** ψ, which is a function in $L^2(\mathbb{R}^d)$, the space of square integrable complex-valued function on $\mathbb{R}^d$. For a particle of mass m, the kinetic energy equals

$$\frac{\hbar^2}{2m}\,(\psi, -\Delta\,\psi) = \frac{\hbar^2}{2m}\int_{\mathbb{R}^d} |\nabla\psi(x)|^2 \mathrm{d}x\,,$$

where $\hbar$ denotes Planck's constant. For convenience, we will set $m = 1/2$ and $\hbar = 1$ in the following. We will also often write $-\Delta = p^2$, with $p = -i\nabla$.

The potential energy in a potential $V(x)$ is

$$(\psi, V\,\psi) = \int_{\mathbb{R}^d} V(x)|\psi(x)|^2 \mathrm{d}x\,.$$

Historically, the most famous uncertainty principle is **Heisenberg**'s: For $\psi \in L^2(\mathbb{R}^d)$ with $(\psi,\psi) = 1$,

$$\big(\psi, p^2\,\psi\big) \geq \frac{d^2}{4}\,\big(\psi, x^2\,\psi\big)^{-1}\,, \tag{1.1}$$

with equality if and only if ψ is a centered Gaussian. This inequality says that ψ can not be localized too close around the origin without the kinetic energy being big. The proof of (1.1) can be found in any standard textbook on quantum mechanics,

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and we shall not repeat it here. It uses the fact that $[p\cdot a, x\cdot b] = -ia\cdot b$ for vectors $a, b \in \mathbb{C}^d$.

Heisenberg's uncertainty principle is not very useful in practice, however. Specifically, while a small value of $(\psi, x^2\psi)$ means that ψ is localized close to the origin, a large value $(\psi, x^2\psi)$ does *not* mean it is spread out. In fact, $(\psi, x^2\psi)$ could be huge even if most of the mass of ψ is localized around the origin, if only the remaining small part of the mass is far away.

A more useful way to quantify the localization properties of ψ around some point (the origin, say) is via Hardy's and Sobolev's inequality, which we shall discuss next.

1.2. Hardy Inequality. Hardy's inequality looks very similar to Heisenberg's given in Eq. (1.1) above, with one important difference: On the right side, on has the expectation value of the inverse of x^2 instead of the inverse of the expectation value of x^2. More precisely, the following holds:

THEOREM 1.1 (**Hardy's inequality**). *For $d \geq 3$,*

$$\left(\psi, p^2\,\psi\right) \geq \frac{(d-2)^2}{4}\left(\psi, \frac{1}{x^2}\,\psi\right). \tag{1.2}$$

The theorem also holds for $d = 1$ if ψ is required to vanish suitably at the origin. With this assumption, it trivially also holds for $d = 2$, of course. Note that $(\psi, x^{-2}\psi) \geq (\psi, x^2\psi)^{-1}$ if $(\psi,\psi) = 1$, by an application of Jensen's inequality and convexity of $x \mapsto x^{-2}$.

The constant $(d-2)^2/4$ in (1.2) is **sharp** in the sense that the inequality with a larger constant is false for some ψ.

PROOF. We ignore some technical details and assume ψ to be sufficiently smooth. We can then write $\psi(x) = |x|^{1-d/2}g(x)$ with g having the property that $g(0) = 0$, for otherwise both sides of the inequality will be infinite. After taking the derivative of ψ and squaring the resulting expression, we obtain

$$\begin{aligned} |\nabla\psi(x)|^2 = {} & \frac{(d-2)^2}{4}\frac{|\psi(x)|^2}{x^2} + |x|^{2-d}|\nabla g(x)|^2 \\ & + (2-d)\Re\,|x|^{1-d}\overline{g(x)}\frac{\partial}{\partial|x|}g(x)\,. \end{aligned} \tag{1.3}$$

The last term vanishes after integration over x, since

$$\Re\,\overline{g(x)}\frac{\partial}{\partial|x|}g(x) = \frac{1}{2}\frac{\partial}{\partial|x|}|g(x)|^2$$

and hence the radial integral over $|x|$ vanishes at every fixed angle. The second term on the right side of (1.3) is strictly positive, and hence we arrive at the desired result. ■

What the proof really showed is that

$$-\Delta - \frac{(d-2)^2}{4x^2} = -|x|^{d/2-1}\nabla|x|^{2-d}\nabla|x|^{d/2-1}\,. \tag{1.4}$$

The term on the right side is positive, which gives Hardy's inequality. It is in fact *strictly* positive, in the sense that it does not have a zero eigenvalue. It annihilates the function $|x|^{1-d/2}$, but this is not a square-integrable function. In particular, (1.2) is strict for any ψ that is not identically zero.

The fact that the constant $(d-2)^2/4$ is sharp can also be easily seen from (1.4). For a (otherwise smooth) function that diverges as $|x|^{1-d/2}$ at the origin the expectation value of $|x|^{-2}$ is infinite while the expectation value of the right side of (1.4) is finite.

In a way similar to Theorem 1.1, one can in fact prove that

$$\int_{\mathbb{R}^d} |\nabla\psi(x)|^p \mathrm{d}x \geq \left(\frac{d-p}{p}\right)^p \int_{\mathbb{R}^d} \frac{|\psi(x)|^p}{|x|^p} \mathrm{d}x \tag{1.5}$$

for any $1 \leq p < d$. One writes $\psi(x) = |x|^{(1-d/p)} g(x)$ and uses the convexity inequality

$$|a+b|^p \geq |a|^p + p|a|^{p-2} \Re\, \overline{a} \cdot b$$

for vectors $a, b \in \mathbb{C}^d$.

In order to take effects of **special relativity** into account, it is useful to consider the kinetic energy to be $\sqrt{p^2+m^2}$ instead of $p^2/(2m)$. By definition,

$$\left(\psi, \sqrt{-\Delta+m^2}\,\psi\right) = \int_{\mathbb{R}^d} |\widehat{\psi}(k)|^2 \sqrt{|2\pi k|^2+m^2}\, \mathrm{d}k$$

where $\widehat{\psi}$ is the **Fourier transform** of ψ, i.e.,

$$\widehat{\psi}(k) = \int_{\mathbb{R}^d} \psi(x)\, \mathrm{e}^{-2\pi i k\cdot x}\, \mathrm{d}x\,.$$

The operator $p = -i\nabla$ thus acts as multiplication by $2\pi k$ in momentum space.

Note that $\sqrt{p^2+m^2} \approx m + p^2/(2m)$ for small $|p|$ (or large m), while $\sqrt{p^2+m^2} \approx |p|$ for large $|p|$ (or small m). In any case,

$$|p| \leq \sqrt{p^2+m^2} \leq |p| + m\,,$$

and for the questions of stability discussed in the following sections one might as well set $m=0$. This is sometimes referred to as the 'ultra-relativistic' limit.

In the case $m=0$, the relativistic kinetic energy has the following nice double integral representation:

Lemma 1.2 (**Integral Representation of Relativistic Kinetic Energy**).

$$\left(\psi, \sqrt{-\Delta}\,\psi\right) = \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^{2d}} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{d+1}} \mathrm{d}x\mathrm{d}y \tag{1.6}$$

Proof. Write $|p| = \lim_{t\to 0} t^{-1}(1-\mathrm{e}^{-t|p|})$ and use the fact that $\mathrm{e}^{-t|p|}$ has the integral kernel

$$\mathrm{e}^{-t|p|}(x,y) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{t}{(t^2+|x-y|^2)^{(d+1)/2}}\,.$$

This leads immediately to (1.6). ■

The analogue of Hardy's inequality (1.2) in the relativistic case is the following. It is also sometimes referred to as Kato's inequality [**17**, Eq. (V.5.33)].

Theorem 1.3 (**Relativistic Hardy Inequality**). *For $d \geq 2$*

$$\left(\psi, \sqrt{-\Delta}\,\psi\right) \geq 2\frac{\Gamma((d+1)/4)^2}{\Gamma((d-1)/4)^2} \left(\psi, \frac{1}{|x|}\,\psi\right) \tag{1.7}$$

As in the non-relativistic case in Theorem 1.1, the constant in (1.7) is sharp. Note that for $d = 3$ it equals $2/\pi$.

Proof. For simplicity, we restrict our attention to the case $d = 3$, which is the most relevant case in view of applications in physics. For general d, the proof works the same way, but the integrals involved are slightly more complicated. See [**15, 31, 11**].

According to Lemma 1.2,

$$(\psi, |p|\,\psi) = \frac{1}{2\pi^2}\int_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^4}\mathrm{d}x\,\mathrm{d}y\,,$$

which we can write as

$$(\psi, |p|\,\psi) = \lim_{\epsilon\to 0}\frac{1}{2\pi^2}\int_{\mathbb{R}^6} \frac{|\psi(x)-\psi(y)|^2}{(|x-y|^2+\epsilon)^2}\mathrm{d}x\,\mathrm{d}y\,. \tag{1.8}$$

The purpose of ϵ is to avoid the singularity at $x = y$. The limit $\epsilon \to 0$ will be taken at the end of the calculation.

We can write

$$|\psi(x)-\psi(y)|^2 = \frac{||x|\psi(x)-|y|\psi(y)|^2}{|x|\,|y|} + |\psi(x)|^2\left(1-\frac{|x|}{|y|}\right) + |\psi(y)|^2\left(1-\frac{|y|}{|x|}\right)\,.$$

Hence the integral in (1.8) can be written as

$$\int_{\mathbb{R}^6} \frac{||x|\psi(x)-|y|\psi(y)|^2}{(|x-y|^2+\epsilon)^2}\frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{|y|} + 2\int_{\mathbb{R}^6}\frac{|\psi(x)|^2}{(|x-y|^2+\epsilon)^2}\left(1-\frac{|x|}{|y|}\right)\mathrm{d}x\,\mathrm{d}y\,.$$

After performing the y angular integration and denoting $t = |y|$, the second integral equals

$$8\pi\int_{\mathbb{R}^3}\mathrm{d}x\,|\psi(x)|^2\int_0^\infty \mathrm{d}t\,\frac{t(t-|x|)}{(|x|^2-t^2)^2+2\epsilon(|x|^2+t^2)+\epsilon^2}\,.$$

The t integral is non-negative and converges, as $\epsilon \to 0$, to the principle-value integral

$$\text{p.v.}\int_0^\infty \frac{t(t-|x|)}{(|x|^2-t^2)^2}\mathrm{d}t = \frac{1}{2|x|}\,.$$

It is tedious but elementary to justify exchanging the x integration and the $\epsilon \to 0$ limit.

We conclude that

$$(\psi,|p|\,\psi) - \frac{2}{\pi}\int_{\mathbb{R}^3}|\psi(x)|^2\frac{1}{|x|}\mathrm{d}x = \lim_{\epsilon\to 0}\frac{1}{2\pi^2}\int_{\mathbb{R}^6}\frac{||x|\psi(x)-|y|\psi(y)|^2}{(|x-y|^2+\epsilon)^2}\frac{\mathrm{d}x}{|x|}\frac{\mathrm{d}y}{|y|}\,.$$

Since the right side is positive, this proves (1.7). ■

The proofs of Theorems 1.3 and 1.1 are very similar. The main idea is to write $\psi(x) = f(x)g(x)$, with $f(x)$ the formal solution of the corresponding variational equation, which is $f(x) = |x|^{1-d/2}$ in the non-relativistic case and $f(x) = |x|^{(1-d)/2}$ in the relativistic case. This procedure is sometimes referred to as **ground state substitution**.

The Hardy inequalities discussed above can be generalized in various ways. One is to kinetic energies of the form $(-\Delta)^s$ for $s > 0$, and this generalization is straightforward. More involved is the L^p generalization (which is (1.5) for $s = 1$), in which case one considers expressions of the form

$$\int_{\mathbb{R}^{2d}} \frac{|\psi(x) - \psi(y)|^p}{|x-y|^{d+ps}}\, \mathrm{d}x\, \mathrm{d}y$$

for general $0 < s < 1$ and $1 \le p < d/s$. The sharp constants for the corresponding Hardy inequalities have only been found very recently in [**13**].

1.3. Sobolev Inequalities. An alternative way to quantify the uncertainty principle in quantum mechanics is **Sobolev's inequality**. Recall the definition of the $L^p(\mathbb{R}^d)$ norms

$$\|\psi\|_p = \left(\int_{\mathbb{R}^d} |\psi(x)|^p \mathrm{d}x \right)^{1/p}$$

for $1 \le p < \infty$, and

$$\|\psi\|_\infty = \operatorname{ess\,sup} |\psi(x)|\,.$$

THEOREM 1.4 (**Sobolev's Inequality**). *For $d \ge 3$*

$$(\psi, -\Delta\, \psi) \ge S_d \|\psi\|^2_{2d/(d-2)} \tag{1.9}$$

with $S_d = d(d-2)|\mathbb{S}^d|^{2/d}/4$. For $d = 2$ one has, for some $S_{2,p} > 0$,

$$(\psi, -\Delta\, \psi) \ge S_{2,p} \|\psi\|_2^{-4/(p-2)} \|\psi\|_p^{2p/(p-2)} \quad \textit{for all } 2 < p < \infty$$

while for $d = 1$

$$(\psi, -\Delta\, \psi) \ge \|\psi\|_2^{-2} \|\psi\|_\infty^4\,.$$

For $d = 1$ and $d \ge 3$ the constants are sharp, while for $d = 2$ the value of the optimal constant $S_{2,p}$ is unknown. We shall skip the proof of this theorem here, and refer the interested reader to [**20**]. Appropriate Sobolev inequalities hold also for fractional powers of $-\Delta$, of course. In the relativistic case, the following holds.

THEOREM 1.5 (**Relativistic Sobolev Inequality**). *For $d \ge 2$,*

$$\left(\psi, \sqrt{-\Delta}\, \psi\right) \ge \widetilde{S}_d \|\psi\|^2_{2d/(d-1)} \tag{1.10}$$

with $\widetilde{S}_d = (d-1)|\mathbb{S}^d|^{1/d}/2$. For $d = 1$ one has, for some $\widetilde{S}_{1,p} > 0$,

$$\left(\psi, \sqrt{-\Delta}\, \psi\right) \ge \widetilde{S}_{1,p} \|\psi\|_2^{-4/(p-2)} \|\psi\|_p^{2p/(p-2)} \quad \textit{for all } 2 < p < \infty.$$

Using the powerful tool of **symmetric-decreasing rearrangement** (see, e.g., [**20**]), one can actually derive Sobolev's inequalities from Hardy's, except for the value of the optimal constants. In this sense, Hardy's inequality is stronger than Sobolev's. The argument goes as follows [**13**].

For a radial, decreasing function ψ,

$$\int_{\mathbb{R}^d} \psi(x)^p \mathrm{d}x = \|\psi\|_p^p \ge \psi(y)^p |y|^d |\mathbb{B}^d|$$

for any $y \in \mathbb{R}^d$. Now take this to the power $1 - 2/p$, multiply by $|\psi(y)|^2 |y|^{d(2/p-1)}$ and integrate over y. This gives

$$\int_{\mathbb{R}^d} \frac{|\psi(y)|^2}{|y|^{d(1-2/p)}} \mathrm{d}x \ge |\mathbb{B}^d|^{1-2/p} \|\psi\|_p^2\,.$$

Hence Sobolev's inequality follows from Hardy's for radial, decreasing functions.

To extend this result to arbitrary functions, the crucial observation is that $(\psi, -\Delta\psi)$ goes down under symmetric-decreasing rearrangement of ψ, while $\|\psi\|_p$ stays the same. Therefore it suffices to prove Sobolev's inequality for symmetric-decreasing functions.

In order to define symmetric-decreasing rearrangement, note first that for any measurable function ψ,

$$|\psi(x)| = \int_0^\infty \chi_{\{|\psi|>t\}}(x)\mathrm{d}t$$

where $\chi_{\{|\psi|>t\}}$ denotes the characteristic function of the set where $|\psi| > t$, i.e.,

$$\chi_{\{|\psi|>t\}}(x) = \begin{cases} 1 & \text{if } |\psi(x)| > t \\ 0 & \text{if } |\psi(x)| \leq t \,. \end{cases}$$

For a general set $A \subset \mathbb{R}^d$, we denote by χ_A^* the characteristic function of a ball of volume $|A|$ centered at the origin. The symmetric-decreasing rearrangement of ψ, denoted by ψ^*, is defined as

$$\psi^*(x) = \int_0^\infty \chi^*_{\{|\psi|>t\}}(x)\mathrm{d}t \,.$$

Note that ψ^* is clearly symmetric-decreasing. From the definition, it is also obvious that $\|\psi^*\|_p = \|\psi\|_p$ for any p since the rearrangement does not change the values of ψ, only the places where these values are taken. What is less obvious is that

$$\left(\psi^*, p^2\, \psi^*\right) \leq \left(\psi, p^2\, \psi\right) \tag{1.11}$$

and we refer to [**20**] for its proof. Inequality (1.11) also holds with p^2 replaced by $|p|$, and hence the argument just given also applies to the relativistic case.

1.4. Consequences for Schrödinger Operators. If V is a (real-valued) potential that goes to zero at infinity, the spectrum of $-\Delta + V$ consists of discrete points in $(-\infty, 0]$ and the continuum $[0,\infty)$. The infimum of the spectrum is called the **ground state energy**, and it is determined by the variational principle

$$E_0 = \inf_{\|\psi\|_2=1} \left(\psi, (-\Delta + V)\,\psi\right) \,.$$

We shall investigate the question for what potentials V the ground state energy is finite.

Recall **Hölder's inequality**, which states that

$$\left|\int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x\right| \leq \|f\|_p\|g\|_q \quad \text{for } 1 \leq p \leq \infty,\, 1/p + 1/q = 1.$$

By combining this with Sobolev's inequality (1.9) we see that, for $d \geq 3$

$$(\psi, -\Delta\,\psi) \geq \frac{S_d}{\|V\|_{d/2}} \left(\psi, |V|\,\psi\right) \,.$$

In particular, $-\Delta + V \geq 0$ if $\|V\|_{d/2} \leq S_d$. That is, $E_0 = 0$ in this case.

More generally, if $V \in L^{d/2} + L^\infty$ then V can be written as $V(x) = w(x) + u(x)$ with $\|w\|_{d/2} \leq S_d$ and u bounded, and hence $E_0 > -\infty$. We leave the demonstration of this fact as an exercise. We can proceed in a similar way for $d \leq 2$ and conclude that E_0 **is finite if**

$$V \in \begin{cases} L^{d/2} + L^\infty & \text{if } d \geq 3 \\ L^{1+\varepsilon} + L^\infty & \text{if } d = 2 \\ L^1 + L^\infty & \text{if } d = 1. \end{cases} \tag{1.12}$$

Similarly, in the relativistic case, the ground state energy of $\sqrt{-\Delta} + V$ is finite if

$$V \in \begin{cases} L^{d} + L^\infty & \text{if } d \geq 2 \\ L^{1+\varepsilon} + L^\infty & \text{if } d = 1. \end{cases} \tag{1.13}$$

As an example, consider the Coulomb potential $V(x) = -|x|^{-1}$ in $d = 3$. Clearly $V \in L^{3/2} + L^\infty$, which explains the stability of the hydrogen atom with non-relativistic kinematics. The relativistic case is borderline, however, since V just fails to be in $L^3 + L^\infty$. We have in fact seen in Subsection 1.2 above that $\sqrt{-\Delta} - \lambda|x|^{-1}$ is bounded from below (in fact, positive) only if $\lambda \leq 2/\pi$.

As this example shows, one can deduce from Hardy's inequality that the singularities in V can actually be slightly stronger than in (1.12) and (1.13) above for E_0 to be finite: In the non-relativistic case $E_0 > -\infty$ if $V(x) \geq -(d-2)^2/(4|x|^2) - C$ or, more generally, if $V(x) \geq -\frac{1}{4}(d-2)^2 \sum_i |x - R_i|^{-2} - C$ for finitely many distinct points $R_i \neq R_j$. We leave the proof of this last statement as an exercise.

After having found conditions on V that guarantee the finiteness of the ground state energy, we will study sums of powers of **all** the negative eigenvalues of Schrödinger operators in the next section.

2. Lieb-Thirring Inequalities

2.1. Introduction. Let $E_0 \leq E_1 \leq E_2 \ldots$ be the **negative eigenvalues** of the Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$, with V satisfying the condition (1.12). **Lieb-Thirring inequalities** concern bounds on the moments

$$\sum_{i \geq 0} |E_i|^\gamma$$

for some $\gamma \geq 0$. The case $\gamma = 0$ corresponds to the **number of negative eigenvalues**.

THEOREM 2.1 (**Lieb-Thirring Inequalities**). *The negative eigenvalues E_i of $-\Delta + V$ satisfy the bounds*

$$\sum_{i \geq 0} |E_i|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma + d/2} \mathrm{d}x \tag{2.1}$$

where $V(x)_- = \max\{-V(x), 0\}$ denotes the negative part of V. The (sharp) values of $\gamma \geq 0$ for which (2.1) holds with $L_{\gamma,d} < \infty$ (independent of V) are

- *for $d = 1$, $\gamma \geq 1/2$* [**23, 30**]
- *for $d = 2$, $\gamma > 0$* [**23**]
- *for $d \geq 3$, $\gamma \geq 0$* [**4, 19, 26, 23**]

The fact that $\gamma > 0$ is necessary for $d = 2$ follows from the fact that $-\Delta - \lambda V$ has a negative eigenvalue for arbitrarily small λ if V is negative. For $d = 1$ this eigenvalue is of the order λ^2 as $\lambda \to 0$, from which one easily deduces that (2.1) can hold only for $\gamma \geq 1/2$ for $d = 1$.

The special (and most difficult) case $\gamma = 0$ for $d \geq 3$ is also known as the **Cwikel-Lieb-Rosenblum** [**4, 19, 26**] bound. Below we will prove (2.1) only for $\gamma > 0$ for $d \geq 2$ and $\gamma > 1/2$ for $d = 1$. Our proof follows the original work by Lieb and Thirring [**23**].

2.2. The Semiclassical Approximation. Another way to write sum of the negative eigenvalues to the power γ is

$$\sum_{i\geq 0} |E_i|^\gamma = \mathrm{Tr}\big(-\Delta + V(x)\big)_-^\gamma \,, \tag{2.2}$$

where Tr denotes the trace. A **semiclassical approximation** of the trace leads to the phase space integral

$$(2\pi)^{-d} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(p^2 + V(x)\big)_-^\gamma \mathrm{d}p\,\mathrm{d}x = L_{\gamma,d}^{\mathrm{scl}} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\,\mathrm{d}x \tag{2.3}$$

where

$$L_{\gamma,d}^{\mathrm{scl}} = (2\pi)^{-d} \int_{|p|\leq 1} (1-p^2)^\gamma \mathrm{d}p = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)} \,.$$

Note that the factor $(2\pi)^{-d}$ in front of the integral in (2.3) is really Planck's constant to the power $-d$; in our units Planck's constant equals 2π.

The reason for calling (2.3) the semiclassical approximation to (2.2) is that

$$\lim_{h\to 0} h^d\, \mathrm{Tr}\big(-h^2\Delta + V(x)\big)_-^\gamma = L_{\gamma,d}^{\mathrm{scl}} \int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2}\,\mathrm{d}x \tag{2.4}$$

under certain assumptions on the potential V. See [**20**, Sect. 12.12] or [**25**, Sect. XIII.15]. Note that because of (2.4) it is necessarily true that $L_{\gamma,d} \geq L_{\gamma,d}^{\mathrm{scl}}$. But when is $L_{\gamma,d} < \infty$, and when does it equal $L_{\gamma,d}^{\mathrm{scl}}$?

2.3. The Sharp Constants. As mentioned above, $L_{\gamma,d} < \infty$ if and only if $\gamma \geq 1/2$ for $d = 1$, $\gamma > 0$ for $d = 2$ and $\gamma \geq 0$ for $d \geq 3$. Some facts about the **sharp values** for $L_{\gamma,d}$ are known:

- $L_{\gamma,d} = L_{\gamma,d}^{\mathrm{scl}}$ for all $\gamma \geq 3/2$ and $d \geq 1$ [**23, 1, 18**]
- $L_{1/2,1} = 1/2$ while $L_{1/2,1}^{\mathrm{scl}} = 1/4$ [**16**]
- $L_{\gamma,d} > L_{\gamma,d}^{\mathrm{scl}}$ if $\gamma < 1$ [**14**]

The optimal constant in the physically most interesting case, $\gamma = 1$ and $d = 3$, remains an **open problem**. It is conjectured to be $L_{1,3} = L_{1,3}^{\mathrm{scl}}$. The best current bound was obtained in [**6**] as

$$L_{1,3} \leq \frac{\pi}{\sqrt{3}} L_{1,3}^{\mathrm{scl}} \,. \tag{2.5}$$

We will use this bound in Section 3 in our proof of Stability of Matter.

2.4. The Birman-Schwinger Principle. We shall now explain the proof of the LT inequalities (2.1). For simplicity we restrict our attention to the non-critical cases, i.e., to $\gamma > 0$ for $d \geq 2$ and $\gamma > 1/2$ for $d = 1$.

From the variational principle for eigenvalues [**20**, Thm. 12.1], it follows that all eigenvalues increase if we replace the positive part of V by zero. Hence, in order to prove (2.1), it suffices to consider the case $V(x) \leq 0$.

The eigenvalue equation

$$-\Delta\psi(x) + V(x)\psi(x) = -e\psi(x)$$

is equivalent to

$$\phi(x) = \int_{\mathbb{R}^d} K_e(x,y)\phi(y)\mathrm{d}y$$

where $\phi(x) = \sqrt{-V(x)}\psi(x)$ and K_e is the **Birman-Schwinger kernel**

$$K_e(x,y) = \sqrt{-V(x)}G_e(x-y)\sqrt{-V(y)}$$

with

$$G_e(x-y) = \int_{\mathbb{R}^d} \frac{1}{(2\pi k)^2 + e}\,\mathrm{e}^{2\pi i k\cdot(x-y)}\,\mathrm{d}k\,.$$

In one dimension, the function G_e equals $G_e(x) = \mathrm{e}^{-\sqrt{e}|x|}/(2\sqrt{e})$, while it is $G_e(x) = \mathrm{e}^{-\sqrt{e}|x|}/(4\pi|x|)$ for $d = 3$.

In other words, $-\Delta + V$ having an eigenvalue $-e < 0$ is equivalent to K_e having an eigenvalue 1; also the multiplicities coincide. This fact is known as the **Birman-Schwinger principle**. The study of the negative eigenvalues of $-\Delta + V$ thus reduces to a study of the spectrum of the family K_e of compact and positive operators.

From the definition it is obvious that K_e is monotone decreasing in e. Hence all the eigenvalues λ_i of K_e are monotone decreasing in e. These are sketched in Figure 1.

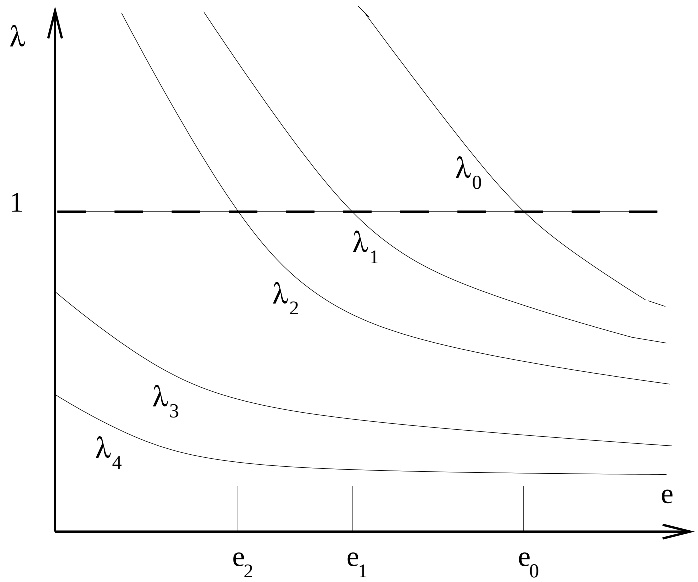

FIGURE 1. Sketch of the eigenvalues of the Birman-Schwinger operator K_e as a function of e. An eigenvalue $-e$ of the Schrödinger operator $-\Delta+V$ is equivalent to K_e having an eigenvalue 1.

From the figure we easily deduce the fact that **the number of eigenvalues of K_e that are ≥ 1 equals the number of eigenvalues of $-\Delta + V$ that are**

$\leq -e$! We shall call this number N_e. For any $m > 0$ we thus have the simply but important inequality

$$N_e \leq \mathrm{Tr}\,(K_e)^m \,. \tag{2.6}$$

Since V is assumed to be non-positive, we can write K_e as the product $K_e = |V|^{1/2} G_e |V|^{1/2}$, with $|V|^{1/2}$ a multiplication operator and G_e a convolution operator. It is a fact that for any two non-negative operators A and B and any $m \geq 1$,

$$\mathrm{Tr}(B^{1/2}AB^{1/2})^m \leq \mathrm{Tr}\, B^{m/2}A^m B^{m/2} \,. \tag{2.7}$$

For the proof for general $m \geq 1$ we refer the reader to [**23, 27, 2, 21**] or to the lecture notes by E. Carlen in this volume. For $m = 2$, however, the proof is very simple. Since $AB - BA$ is anti-hermitian, its square is non-positive, i.e., $(AB - BA)^2 \leq 0$. Using this and the cyclicity of the trace one concludes that

$$\mathrm{Tr}\, ABAB \leq \mathrm{Tr}\, A^2B^2 \,.$$

The case $m = 2$ is actually the one needed to prove the LT inequality in the physically relevant case $\gamma = 1$ and $d = 3$, as we shall see below. If one is only interested in this special case, one does not need the general bound (2.7).

From (2.6) and (2.7) we conclude that, for $m \geq 1$,

$$\begin{aligned} N_e &\leq \mathrm{Tr}\, |V|^{m/2} (G_e)^m |V|^{m/2} \\ &= \left(\int_{\mathbb{R}^d} \frac{1}{((2\pi k)^2 + e)^m} \mathrm{d}k \right) \int_{\mathbb{R}^d} |V(x)|^m \mathrm{d}x \,. \end{aligned}$$

The k integral is finite if $m > d/2$. In fact, it is given by

$$\int_{\mathbb{R}^d} \frac{1}{((2\pi k)^2 + e)^m} \mathrm{d}k = e^{-m+d/2} \frac{\Gamma(m - d/2)}{(4\pi)^{-d/2}\Gamma(m)} =: e^{-m+d/2} C_{d,m} \,.$$

Hence we have the upper bound

$$N_e \leq C_{d,m} e^{-m+d/2} \int_{\mathbb{R}^d} |V(x)|^m \mathrm{d}x$$

for any m satisfying the conditions $m \geq 1$ and $m > d/2$.

To obtain information on the sum of powers of negative eigenvalues of $-\Delta + V$, note that

$$\sum_{i \geq 0} |E_i|^\gamma = \gamma \int_0^\infty e^{\gamma - 1} N_e \mathrm{d}e \,. \tag{2.8}$$

Using the above bound on N_e, the e integral diverges, however, either at 0 or at ∞. As a way out, consider

$$W_e(x) = \max\{-V(x) - e/2, 0\} \,.$$

Then

$$N_e(V) = N_{e/2}(V + e/2) \leq N_{e/2}(-W_e)$$

and hence

$$\begin{aligned} \sum_{i \geq 0} |E_i|^\gamma &\leq \gamma C_{d,m} \int_{\mathbb{R}^d} \int_0^{-2V(x)} e^{\gamma - 1 - m + d/2} (-V(x) - e/2)^m \,\mathrm{d}e \,\mathrm{d}x \\ &= C'_{\gamma,d,m} \int_{\mathbb{R}^d} |V(x)|^{\gamma + d/2} \mathrm{d}x \,. \end{aligned}$$

For $C'_{\gamma,d,m}$ to be finite we need $d/2 < m < \gamma + d/2$, i.e., $\gamma > 0$ and $\gamma > 1/2$ for $d = 1$. A possible choice is $m = (\gamma + d)/2$. This completes the proof. ■

2.5. Possible Extensions. LT inequalities are known for a more general class of operators. Extensions that are important for applications in physics are:

- **Magnetic fields:** One can replace $-\Delta$ by $-(\nabla - iA(x))^2$ for a real-valued vector-potential A (whose curl is the magnetic field). Recall the **diamagnetic inequality**, which states that
$$\big(\psi, -(\nabla - iA(x))^2\,\psi\big) \geq (|\psi|, -\Delta\,|\psi|)\ .$$
Hence the *lowest* eigenvalue E_0 always goes up when a vector field is introduced, but not necessarily the sum of powers of the eigenvalues. Hence the extension of the LT inequalities to magnetic fields is non-trivial.
- **Fractional Schrödinger operators:** One can use $\sqrt{-\Delta}$ instead of $-\Delta$ for the kinetic energy [**5**]. The appropriate L^p-norm of the potential is determined by semiclassics as
$$\operatorname{Tr}\left(\sqrt{-\Delta} + V\right)_-^\gamma \leq K_{\gamma,d}\int V(x)_-^{\gamma+d}\mathrm{d}x\,.$$
A similar result holds for $(-\Delta)^s$ for arbitrary $s > 0$ and appropriate γ.

2.6. Kinetic Energy Inequalities. For N particles satisfying **Fermi-Dirac statistics** (e.g., electrons), the wave functions have to be antisymmetric, i.e.,
$$\psi(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_N) = -\psi(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_N)$$
for every $1 \leq i \neq j \leq N$. (We ignore spin for simplicity.) We leave it as an exercise to show that for any such ψ with $(\psi,\psi) = 1$,
$$\left(\psi, \textstyle\sum_{j=1}^N \left[-\Delta_j + V(x_j)\right]\,\psi\right) \geq \sum_{i=0}^{N-1} E_i \tag{2.9}$$
with E_j the negative eigenvalues of $-\Delta + V$. The LT inequality for $\gamma = 1$ implies that the latter sum is bounded from below by
$$-L_{1,d}\int_{\mathbb{R}^d} V(x)_-^{1+d/2}\mathrm{d}x\,. \tag{2.10}$$

For a given (antisymmetric) ψ, let ϱ_ψ denote its one-particle density, i.e.,
$$\varrho_\psi(x) = N\int_{\mathbb{R}^{d(N-1)}} |\psi(x, x_2,\ldots,x_N)|^2\mathrm{d}x_2\cdots\mathrm{d}x_N\,.$$
If we choose
$$V(x) = -c\,\varrho_\psi(x)^{2/d}$$
for some $c > 0$, we conclude from (2.9)–(2.10) that
$$\left(\psi, -\textstyle\sum_{j=1}^N \Delta_j\,\psi\right) \geq \left(c - L_{1,d}c^{1+d/2}\right)\int_{\mathbb{R}^d}\varrho_\psi(x)^{1+2/d}\mathrm{d}x\,.$$
To make the right side as large as possible, the optimal choice of c is $c = [L_{1,d}(1 + d/2)]^{-2/d}$. This yields
$$\left(\psi, -\textstyle\sum_{j=1}^N \Delta_j\,\psi\right) \geq \frac{d}{d+2}\left(\frac{2}{L_{1,d}(d+2)}\right)^{2/d}\int_{\mathbb{R}^d}\varrho_\psi(x)^{1+2/d}\mathrm{d}x\,. \tag{2.11}$$

Inequality (2.11) can be viewed as an **uncertainty principle** for many-particle systems. We emphasize that the antisymmetry of ψ is essential for (2.11) to hold with an N-independent constant on the right side. For general ψ, (2.11) holds only if the right side is multiplied by $N^{-2/d}$.

Inequality (2.11) is **equivalent** to the LT inequality (2.1) for $\gamma = 1$ in the sense that validity of (2.11) for all antisymmetric ψ implies (2.1) with the corresponding constant $L_{1,d}$. The demonstration of this fact is left as an exercise.

The right side of (2.11), with $L_{1,d}$ replaced by $L_{1,d}^{\mathrm{scl}}$, is just the **semiclassical approximation** to the kinetic energy of a many-body system. To see this, let us calculate the sum of the lowest N eigenvalues of the Laplacian on a cube of side length ℓ. For large N, boundary conditions are irrelevant, and hence we can use periodic boundary conditions in which case the eigenvalues of $-\Delta$ are just $(2\pi k)^2$ with $k \in (\mathbb{Z}/\ell)^d$. Replacing sums by integrals the sum of the lowest N eigenvalues is

$$(2\pi)^2 \ell^d \int_{|k|\leq\mu} |k|^2 \mathrm{d}k \tag{2.12}$$

with μ determined by

$$\ell^d \int_{|k|\leq\mu} \mathrm{d}k = N\,.$$

A simple computation thus shows that for this value of μ (2.12) equals

$$\frac{d}{d+2}\left(\frac{2}{L_{1,d}^{\mathrm{scl}}(d+2)}\right)^{2/d} \ell^d \left(\frac{N}{\ell^d}\right)^{1+2/d}. \tag{2.13}$$

In a semiclassical approximation, one can estimate the lowest energy of a system with particle density $\varrho(x)$ by (2.13) replacing N/ℓ^d by $\varrho(x)$ and integrating over x instead of multiplying by ℓ^d. One indeed arrives at the right side of (2.11) this way, except for the prefactor.

3. Application: The Stability of Matter

3.1. Introduction. Ordinary matter composed of electrons and nuclei is described by the **Hamiltonian**

$$\begin{aligned} H = &-\sum_{i=1}^N \Delta_i - \sum_{i=1}^N\sum_{j=1}^M \frac{Z}{|x_i - R_j|} \\ &+ \sum_{1\leq i<j\leq N} \frac{1}{|x_i-x_j|} + \sum_{1\leq k<l\leq M} \frac{Z^2}{|R_k-R_l|}\,. \end{aligned} \tag{3.1}$$

The nuclei have charge Z and are located at positions $R_j \in \mathbb{R}^3$, $j = 1,\ldots,M$, which are treated as parameters. The electron coordinates are $x_i \in \mathbb{R}^3$, $i = 1,\ldots,N$, and Δ_i denotes the Laplacian with respect to x_i. Since elections are fermions, the wave functions $\psi(x_1,\ldots,x_N)$ are **antisymmetric** functions of $x_i \in \mathbb{R}^3$. (For simplicity, we ignore spin here.)

Let $E_0(N,M)$ denote the **ground state energy** of H, i.e.,

$$E_0(N,M) = \inf_{\{R_j\}} \inf_{(\psi,\psi)=1} (\psi, H\,\psi)\,.$$

Note that besides minimizing over electron wavefunctions ψ, we also minimize over all nuclear coordinates R_j, $j = 1, \ldots, M$. Everyday experience tells us that

$$E_0(N, M) \geq -c(N + M)$$

for some $c > 0$. If not, you would not be reading this notes! If the energy would decrease faster than linearly with the particle number, bulk matter would not be stable but rather implode; adding two half-filled glasses of water would release a huge amount of energy.

Why is the energy bounded from below by $N+M$? After all, there are $(N+M)^2$ terms in the Hamiltonian, NM of which are negative. In other words, **why is matter stable?** The antisymmetry of the allowed wave functions ψ turns out to be crucial. Without it, $E_0(N, M)$ would decease as $-C \min\{N, M\}^{5/3}$ for large particle number.

Stability of Matter was first proved by Dyson and Lenard in 1967 [**7**]. In 1975, Lieb and Thirring [**22**] gave a very elegant proof, using the kinetic energy inequalities derived in Section 2.6 and the "no-binding theorem" in Thomas-Fermi theory.

We shall follow a different route to prove Stability of Matter (due to Solovej [**28**]) which is probably the shortest. One uses

- The Lieb-Thirring inequality for $d = 3$ and $\gamma = 1$.
- Baxter's electrostatic inequality

In the former, the antisymmetry of the wave functions enters. The latter is a pointwise bound on the total Coulomb potential and has nothing to do with quantum mechanics. We shall explain it next.

3.2. Baxter's electrostatic inequality. Baxter [**3**] proved the following in 1980.

Theorem 3.1 (**Baxter's Electrostatic Inequality**). *For any $x_i, R_j \in \mathbb{R}^3$, $i = 1, \ldots, N$, $j = 1, \ldots M$, and any $Z > 0$,*

(3.2)
$$-\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{Z}{|x_i - R_j|} + \sum_{1\leq i<j\leq N} \frac{1}{|x_i - x_j|} + \sum_{1\leq k<l\leq M} \frac{Z^2}{|R_k - R_l|} \geq -\sum_{i=1}^{N} \frac{2Z+1}{\mathcal{D}(x_i)}$$

where $\mathcal{D}(x) = \min_j |x - R_j|$ denotes the distance to the nearest nucleus.

This inequality quantifies **electrostatic screening**. Effectively, an electron sees only the nearest nucleus. The coefficient $2Z + 1$ on the right side is not sharp. Inequality (3.1) was later improved by Lieb and Yau [**24**] to yield the sharp value Z for x close to one of the nuclei.

The proof of Theorem 3.1 given here does not follow the original one in [**3**]. We shall instead follow a suggestion by Solovej (private communication) and deduce it from a Lemma by Lieb and Yau [**24**]. The argument uses essentially only two things:

- The Coulomb potential $1/|x|$ has positive Fourier transform
- **Newton's theorem**, which states that
$$\int_{\mathbb{R}^3} \frac{f(x)}{|x-y|}\mathrm{d}x = \frac{1}{|y|}\int_{|x|\leq|y|} f(x)\mathrm{d}x + \int_{|x|\geq|y|} \frac{f(x)}{|x|}\mathrm{d}x$$
for any *radial* function f.

Proof of Theorem 3.1. For a measure $\mu(\mathrm{d}x)$, let $D(\mu)$ denote the electrostatic energy

$$D(\mu) = \frac{1}{2}\int_{\mathbb{R}^6} \frac{1}{|x-y|}\mu(\mathrm{d}x)\mu(\mathrm{d}y)\,.$$

Note that $D(\mu) \geq 0$ even if μ is not positive; this follows from the fact the $|x|^{-1}$ has a positive Fourier transform. Let

$$\phi(x) = -\frac{Z}{\mathcal{D}(x)} + \sum_{j=1}^{M} \frac{Z}{|x-R_j|} \tag{3.3}$$

denote the Coulomb attraction to all but the nearest nuclei. We shall first show that

$$D(\mu) - \int \phi(x)\mu(\mathrm{d}x) + \sum_{1\leq k<l\leq M} \frac{Z^2}{|R_k - R_l|} \geq 0 \tag{3.4}$$

for any measure μ.

The function ϕ is harmonic, i.e., $\Delta\phi = 0$, except on the surfaces $\{x : |x-R_k| = |x-R_j| \text{ for some } k \neq j\}$. An explicit computation shows that ϕ is superharmonic on all of $\mathbb{R}^3$, i.e.,

$$-\Delta\phi(x) = 4\pi\,\nu(\mathrm{d}x) \tag{3.5}$$

for some non-negative measure ν which is supported on these surfaces. Using $D(\mu-\nu) \geq 0$ we have

$$D(\mu) - \int \phi(x)\mu(\mathrm{d}x) = D(\mu-\nu) - D(\nu) \geq -D(\nu)\,.$$

It remains to calculate $D(\nu)$. Using (3.3) and (3.5) it is not difficult to see that

$$\begin{aligned} D(\nu) &= \frac{1}{2}\int \phi(x)\nu(\mathrm{d}x) \\ &= \sum_{1\leq k<l\leq M} \frac{Z^2}{|R_k-R_l|} - \frac{1}{2}\int_{\mathbb{R}^3} \frac{Z}{\mathcal{D}(x)}\nu(\mathrm{d}x)\,. \end{aligned}$$

Since $\nu \geq 0$ the last expression is negative and can be dropped for an upper bound. This proves (3.4).

For simplicity, denote $d_i = \mathcal{D}(x_i)$, and let $\mu_i(\mathrm{d}x)$ be the normalized uniform measure supported on a sphere of radius $d_i/2$ centered at x_i, i.e., $\mu_i(\mathrm{d}x) = (d_i^2\pi)^{-1}\delta(|x-x_i| - d_i/2)\mathrm{d}x$. We shall use (3.4) with $\mu = \sum_{i=1}^N \mu_i$. If we replace the electron point charges by the smeared out spherical charges μ_i, the electrostatic interaction among the electrons is reduced because the interaction energy between two spheres is less than or equal to that between two points. This follows from Newton's Theorem, which also implies that the interaction between the smeared electrons and the nuclei is not changed, since the radius $d_i/2$ is less than the distance of x_i to any of the nuclei. Hence

$$\begin{aligned}&\sum_{1\le i<j\le N}\frac{1}{|x_i-x_j|}-\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{Z}{|x_i-R_j|}\\ &\ge \sum_{1\le i<j\le N}\int_{\mathbb{R}^6}\frac{1}{|x-y|}\mu_i(\mathrm{d}x)\mu_j(\mathrm{d}y)-\sum_{j=1}^{M}\int_{\mathbb{R}^3}\frac{Z}{|x-R_j|}\mu(\mathrm{d}x)\\ &= D(\mu)-\sum_{j=1}^{M}\int_{\mathbb{R}^3}\frac{Z}{|x-R_j|}\mu(\mathrm{d}x)-\sum_{i=1}^{N}\frac{1}{d_i}\,.\end{aligned}$$

We have used that $D(\mu_i)=1/d_i$. An application of (3.4) yields the lower bound

$$\begin{aligned}&-\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{Z}{|x_i-R_j|}+\sum_{1\le i<j\le N}\frac{1}{|x_i-x_j|}+\sum_{1\le k<l\le M}\frac{Z^2}{|R_k-R_l|}\\ &\ge -\sum_{i=1}^{N}\left(\int_{\mathbb{R}^3}\frac{Z}{\mathcal{D}(x)}\mu_i(\mathrm{d}x)+\frac{1}{d_i}\right)\,.\end{aligned}$$

To finish the proof it suffices to show that for x in the support of μ_i, we have $\mathcal{D}(x)=\min_j|x-R_j|\ge d_i/2$. This follows from the triangle inequality, which implies that for any k

$$|x-R_k|\ge|x_i-R_k|-|x_i-x|\ge d_i-d_i/2=d_i/2\,.$$

In the last step, we used that $|x_i-R_k|\ge d_i$ by definition, and $|x_i-x|=d_i/2$ for x on the sphere centered at x_i. ■

3.3. Proof of Stability of Matter. The proof of Stability of Matter for the Hamiltonian (3.1) is an easy consequence of (2.9), Theorem 2.1 for $d=3$ and $\gamma=1$, as well as Theorem 3.1.

Using Baxter's electrostatic inequality (3.2), the Hamiltonian (3.1) is bounded from below by

$$H\ge\sum_{i=1}^{N}\left[-\Delta_i-\frac{2Z+1}{\mathcal{D}(x_i)}\right]\,.$$

The right side is just a sum of independent one-particle terms! For **antisymmetric** functions, it is bounded below by the sum of the lowest N eigenvalues of $-\Delta-(2Z+1)/\mathcal{D}(x)$. We can pick a $\mu>0$ and use the LT inequality (2.1) for $d=3$ and $\gamma=1$ to obtain

$$H\ge-\mu N-L_{1,3}\int_{\mathbb{R}^3}\left[\mu-\frac{2Z+1}{\mathcal{D}(x)}\right]_-^{5/2}\mathrm{d}x\,.$$

The integral is bounded by

$$\begin{aligned}\int_{\mathbb{R}^3}\left[\mu-\frac{2Z+1}{\mathcal{D}(x)}\right]_-^{5/2}\mathrm{d}x&\le M\int_{\mathbb{R}^3}\left[\mu-\frac{2Z+1}{|x|}\right]_-^{5/2}\mathrm{d}x\\ &=M\frac{5\pi^2}{4}\frac{(2Z+1)^3}{\sqrt{\mu}}\,.\end{aligned}$$

We can now optimize over μ. If we use, in addition, the bound (2.5) on the LT constant $L_{1,3}$, we have completed the proof of

Theorem 3.2 (**Stability of Matter**). *On the space of antisymmetric functions, the Hamiltonian (3.1) is bounded from below by*

$$H \geq -\frac{\pi^{2/3}}{4}(2Z+1)^2\, M^{2/3}N^{1/3}\,.$$

Since $M^{2/3}N^{1/3} \leq N/3 + 2M/3$, this yields the desired *linear* lower bound on the ground state energy.

3.4. Further Results on Stability of Matter. Since the work of Lieb and Thirring, stability of matter has been proved for many more models, including

- **Relativistic kinematics**, where $\sqrt{-\Delta}$ replaces $-\Delta$ in the kinetic energy
- **Magnetic fields** and their interaction with the **electron spin**
- models with kinetic energy described by the **Dirac operator**
- certain approximations to **quantum electrodynamics**, where the electromagnetic field is quantized.

For details and a pedagogical presentation of this subject, we refer the reader to [**21**].

4. Hardy-Lieb-Thirring Inequalities

Recall **Hardy's inequalities** (1.2) and (1.7). For general fractional powers of the Laplacian, they read

$$\left(\psi, \left[(-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s}\right]\, \psi\right) \geq 0 \tag{4.1}$$

for $0 < s < d/2$ and $\psi \in L^2(\mathbb{R}^d)$. The sharp constant $\mathcal{C}_{s,d}$ is known to be [**15**]

$$\mathcal{C}_{s,d} = 2^{2s}\frac{\Gamma((d+2s)/4)^2}{\Gamma((d-2s)/4)^2}$$

but there is no optimizer for this inequality. The variational equation is satisfied for $\psi(x) = |x|^{s-d/2}$.

Questions:

- Can (4.1) be improved by adding a Sobolev term $\|\psi\|_q^2$ on the right side?
- Do Lieb-Thirring inequalities hold for $(-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s} + V$ (in terms of $\|V\|_q$ alone)?

Although the second question seems stronger than the first, these two are actually intimately related. For an investigation of this relation in a more abstract setting we refer the reader to [**12**]. Both answers are Yes, as was proved in [**11**]. A simpler proof was obtained by Frank in [**9**], using an elegant Lemma of Solovej, Sørensen, and Spitzer [**29**]. We will follow the latter route here.

4.1. Hardy Inequalities with Remainder. The following theorem is the key to understanding the answers to the questions just raised.

Theorem 4.1 (**Hardy with Remainder**). *For $0 < t < s < d/2$, there exists a $\kappa_{d,s,t} > 0$ such that*

$$\left(\psi, \left[(-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s}\right]\, \psi\right) \geq \kappa_{d,s,t}\left(\psi, (-\Delta)^t\, \psi\right)^{s/t}\|\psi\|_2^{2(1-s/t)} \tag{4.2}$$

for all $\psi \in L^2(\mathbb{R}^d)$.

The proof of this theorem can be found in [**9**]. In the special case $d = 3$ and $s = 1/2$ it was obtained earlier in [**29**]. We only give a sketch of the proof here, and refer to [**9, 29**] for details.

Sketch of Proof. In terms of the Fourier transform, the Hardy potential can be written as

$$\int_{\mathbb{R}^d} |\psi(x)|^2|x|^{-2s}\mathrm{d}x = \pi^{2s-d/2}\frac{\Gamma(d/2-s)}{\Gamma(s)}\int_{\mathbb{R}^{2d}} \frac{\overline{\widehat{\psi}(k)}\widehat{\psi}(q)}{|k-q|^{d-2s}}\,\mathrm{d}k\,\mathrm{d}q\,.$$

A simple Schwarz inequality shows that, for any positive function h on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^{2d}} \frac{\overline{\widehat{\psi}(k)}\widehat{\psi}(q)}{|k-q|^{d-2s}}\,\mathrm{d}k\,\mathrm{d}q \le \int_{\mathbb{R}^d} t_h(k)|\widehat{\psi}(k)|^2\mathrm{d}k\,,$$

where

$$t_h(k) = \frac{1}{h(k)}\int_{\mathbb{R}^d}\frac{h(q)}{|k-q|^{d-2s}}\mathrm{d}q\,.$$

Hardy's inequalities (4.1) follow from this by choosing $h(k) = |k|^{-s-d/2}$. The improvement (4.2) can be obtained by choosing instead $h(k) = |k|^{-s-d/2}(1+\lambda|k|^{2(t-s)})^{-1}$ for $\lambda > 0$ and optimizing over the value of λ. ■

4.2. Consequences: Hardy-Sobolev and Hardy-Lieb-Thirring Inequalities. We can now combine Theorem 4.1 with the Sobolev inequality

$$\big(\psi, (-\Delta)^t\,\psi\big) \ge S_{t,d}\|\psi\|_q^2\,, \quad q = 2d/(d-2t)$$

which is the generalization of (1.9)–(1.10) to general fractional Laplacians. This yields

Corollary 4.2 (**Hardy-Sobolev Inequalities**). *For $\psi \in L^2(\mathbb{R}^d)$, $0 < s < d/2$ and $2 < q < 2^* := 2d/(d-2s)$,*

$$\big(\psi, \big[(-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s}\big]\ \psi\big) \ge S_{d,s,q}\|\psi\|_q^{2(1+\sigma)}\|\psi\|_2^{-2\sigma}$$

with $\sigma = 2(1-q/2^*)/(q-2) > 0$.

The value of σ is determined by scale invariance. Note that q is assumed to be **strictly smaller** than the critical Sobolev exponent $2^* = 2d/(d-2s)$. One can, in fact, show that $S_{d,s,q} \to 0$ as $q \to 2^*$.

Given the fact that Lieb-Thirring inequalities hold for $(-\Delta)^t + V$, the improved Hardy inequality of Theorem 4.1 implies

Corollary 4.3 (**Hardy-Lieb-Thirring Inequalities**). *For some constants $0 < M_{\gamma,d,s} < \infty$,*

$$\mathrm{Tr}\big((-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s} + V(x)\big)_-^\gamma \le M_{\gamma,d,s}\int_{\mathbb{R}^d} V(x)_-^{\gamma+d/2s}\,\mathrm{d}x \tag{4.3}$$

for $\gamma > 0$ and $0 < s < d/2$.

A sketch of the proof of Corollary 4.3 is given below. The special case $s = 1$ for $d \ge 3$ was proved earlier in [**8**].

It is not difficult to see that $\gamma > 0$ is **necessary**. An arbitrarily small non-positive potential V will always create a bound state of the operator $(-\Delta)^s - \mathcal{C}_{s,d}|x|^{-2s} + V(x)$!

In order to prove (4.3), one uses again (2.8) to reduce the question to bounding the number of eigenvalues less or equal to e. One then uses Theorem 4.1 together with the simple convexity bound

$$\left(\psi, (-\Delta)^t \psi\right)^{s/t} \|\psi\|_2^{2(1-s/t)} \geq \frac{s}{t} \left(\frac{e}{4(s/t-1)}\right)^{1-t/s} \left(\psi, (-\Delta)^t \psi\right) - \frac{e}{4}\|\psi\|_2^2$$

and proceeds as in the proof of Theorem 2.1. We refer to [**9**] for the details.

For $s \leq 1$, the inequalities (4.3) also hold in the magnetic case, i.e., with $(-\Delta)^s$ replaced by $|\nabla - iA(x)|^{2s}$, with constants independent of A [**11, 9**]. This is relevant for applications to (pseudo-)relativistic models in physics, as we shall discuss below.

4.3. Application: Stability of Relativistic Matter. As in Section 3, we now consider a quantum-mechanical model of N electrons and M fixed nuclei. Besides the electrostatic Coulomb interaction, the electrons interact with an arbitrary external magnetic field $B = \operatorname{curl} A$, in which case p has to be replaced by $p - A$ in the kinetic energy. To take effects of **special relativity** into account, we shall also replace the kinetic energy operator $-(\nabla - iA)^2$ by the corresponding (pseudo-)relativistic expression

$$\sqrt{-(\nabla_j - iA)^2 + m^2} - m\,.$$

Note that, for large m, this reduces to $-(\nabla - iA)^2/(2m)$ to leading order.

The resulting many-particle **Hamiltonian** is then

$$H_{N,M} = \sum_{j=1}^{N} \left(\sqrt{-(\nabla_j - iA(x_j))^2 + m^2} - m\right) + \alpha V_{N,M}\,,$$

with $V_{N,M}$ the total electrostatic potential,

$$V_{N,M}(x_1, \dots, x_N; R_1, \dots R_M)$$
$$= \sum_{1\leq i<j\leq N} \frac{1}{|x_i - x_j|} - \sum_{j=1}^{N}\sum_{k=1}^{M} \frac{Z}{|x_j - R_k|} + \sum_{1\leq k<l\leq M} \frac{Z^2}{|R_k - R_l|}\,.$$

The constant $\alpha > 0$ in front of $V_{N,M}$ is the **fine structure constant**, which equals $\alpha = 1/137$ in nature. The reason it did not appear in the non-relativistic model (3.1) is that in the latter case it can simply be removed by scaling. Not so in the relativistic case; in fact, since the constant in the relativistic Hardy inequality (1.7) is sharp (and equal to $2/\pi$ for $d = 3$) we see that it is necessary that $Z\alpha \leq 2/\pi$ for $H_{N,M}$ to be bounded from below, even for $N = M = 1$.

The Pauli exclusion principle dictates that $H_{N,M}$ acts on **antisymmetric** functions $\psi(x_1, \dots, x_N)$. **Stability of matter** means that on this space

$$H_{N,M} \geq -\text{const.}\,(N + M)$$

with a constant that is independent of the positions R_k of the nuclei and of the magnetic vector potential A. In the relativistic case, one observes that, by scaling, either $\inf_{R_k,A}(\inf \operatorname{spec} H_{N,M}) \geq -mN$ or $\inf_{R_k,A}(\inf \operatorname{spec} H_{N,M}) = -\infty$.

The following theorem was proved in [**10**].

THEOREM 4.4 (**Stability of Relativistic Matter with Magnetic Fields**). *Assume that $Z\alpha \leq 2/\pi$ and $\alpha \leq 1/133$. Then (on the space of antisymmetric wave functions)*

$$H_{N,M} \geq -mN$$

for all values of the nuclear coordinates R_j and the magnetic vector potential A.

Theorem 4.4 was proved by Lieb and Yau [**24**] in the non-magnetic case $A = 0$, but the more general case of non-zero A was open. One of the key ingredients in the proof in [**10**] is the Hardy-Lieb-Thirring inequality (4.3) (more precisely, its generalization to include arbitrary magnetic vector potentials) in the case $\gamma = 1$, $d = 3$ and $s = 1/2$. In order to get the explicit bound on the allowed values of α, which barely covers the physical case $\alpha = 1/137$, one needs a good bound on the relevant constant $M_{1,3,1/2}$. The ones currently available are not good enough to cover the physical case. In [**10**] only the special case of V being constant inside a ball centered at the origin and $+\infty$ outside the ball was needed, however. In this special case, a good enough bound on the corresponding constant was obtained.

Lieb and Yau also showed that $\inf \operatorname{spec} H_{N,M} = -\infty$ if $\alpha > \alpha_c$ for some some $\alpha_c > 0$ independent of Z. Hence a bound on α is indeed needed!

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