On the Static and Dynamical Collapse of White Dwarfs

Christian Hainzl

ABSTRACT. We give an overview over the subject of the stellar collapse of white dwarfs, starting with a brief history starting from Chandrasekhar's original approach till the latest developments concerning the dynamical collapse in terms of finite time blow up of the associated Hartree-Fock equation.

1. Introduction

At the age of 19, on his way from India to England, S. Chandrasekhar calculated the critical mass M_C of cold stars, such as white dwarfs, beyond which gravitational collapse can occur. For this work along with his extensive studies of solar objects [**1**] he was later awarded the Nobel prize for. Actually, the fact that relativistic kinematics weakens the quantum mechanics energy to such an extent that massive objects, like white dwarfs, may have a limiting mass was independently realized by several other people, I. Frenkel, E. Stoner, and L. D. Landau. Two things are essentially important for the understanding of white dwarfs. Special relativity and the Pauli-principle. Together, of course, with the fact that white dwarfs are ionized to a high degree and consequently establish a *charge neutrality* to such extent that the Coulomb interaction appears to be negligible. Let us look at Chandrasekhars original argument more closely. His calculation was based on the thermodynamic principle that, for systems in equilibrium, the gravitational pressure has to be balanced by the pressure of the corresponding matter, i.e.

$$P_{\text{mat}} = P_{\text{grav}}.$$

To obtain the gravitational pressure of a radial star with density $\rho(r)$, Chandrasekhar used Newton's inverse square law of gravitation, which says that the mass

$$dm = \rho(r) r^2 dr d\Omega,$$

1991 *Mathematics Subject Classification.* Primary 81V17; Secondary 82C22.
Key words and phrases. Gravitational collapse, Chandrasekhar limit, Hartree-Fock evolution equation.

sitting at a distance r to the center of the star, and inside the volume $dV = r^2 dr d\Omega$, feels the gravitational force

$$dF_{\text{grav}} = -G\frac{M(r)dm}{r^2} = -G\frac{M(r)\rho(r)r^2 dr d\Omega}{r^2},$$

where

$$M(r) = 4\pi \int_0^r \rho(s)s^2 ds$$

is the mass of the star inside the radius r, and G is the proportionality factor, known as the gravitational constant. In the derivation of the equation he also used Newton's theorem stating that radial masses act on objects as if their mass was centered at its origin. The gravitational pressure henceforth satisfying

$$dP_{\text{grav}} = \frac{dF_{\text{grav}}}{dA} = \frac{dF_{\text{grav}}}{r^2 d\Omega} = -G\frac{M(r)\rho(r)}{r^2}dr, \tag{1.1}$$

allows the equation for the gravitational-hydrostatic equilibrium to be rewritten as

$$\dot{P}_{\text{mat}} = -G\frac{\rho(r)M(r)}{r^2}. \tag{1.2}$$

Chandrasekhar further made the assumption that a cold star under very high pressure is best described by a degenerate Fermi-gas of relativistic particles at temperature $T = 0$. The corresponding equation of state, see equation (4.5) in the appendix, is of the form $P_{\text{mat}}(\rho) \sim \rho^{4/3}$. Notice, that this choice for the equation of state, for P_{mat}, is the *only* place where quantum mechanics entered the calculations of Chandrasekhar. Plugging into (1.2) led him to the so called Lane-Emden equation of index 3 and to the critical mass

$$M_c \approx 0.77 m_n^{-2} G^{-3/2} \approx 1.4 M_\odot,$$

where $M_\odot$ denotes the solar mass, and m_n is the mass of the nucleon

Let us be here a bit more precise. Solving equation (1.2) for $M(r)$,

$$-GM(r) = \frac{\dot{P}_{\text{mat}} r^2}{\rho(r)},$$

and differentiating with respect to r, leads to the second order ODE,

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\,P_{\text{mat}}}{dr}\right) = -4\pi G\rho r^2. \tag{1.3}$$

Using now the equation of state for a degenerate extreme relativistic Fermi-gas, see equation (4.5),

$$P_{\text{mat}} = K\rho^{4/3}, \quad \text{with} \quad K = \frac{1}{12\pi^2}\left(\frac{3\pi^2}{2m_n}\right)^{4/3},$$

the equation (1.3) takes the form

$$K\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho^{4/3}}{dr}\right) = -4\pi G\rho r^2, \tag{1.4}$$

together with the natural initial conditions

$$\rho(0) = \rho_0, \quad \rho'(0) = 0.$$

Defining the dimensionless quantities

$$\Theta = \left(\frac{\rho}{\rho_0}\right)^{1/3}, \quad x = \left[\frac{\pi G}{K}\right]^{1/2} \rho_0^{1/3} r, \tag{1.5}$$

the equation (1.4) can be rewritten as

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\Theta(x)}{dx}\right) = -\Theta^3, \tag{1.6}$$

with initial conditions

$$\Theta(0) = 1, \quad \Theta'(0) = 0.$$

The corresponding solution is known as the *Lane-Emden-function* of index 3. Observe that (1.5) gives a relation between the total radius R and the initial density ρ_0, via the radius x_r of the compactly supported Lane-Emden-function, which is computed numerically as $x_r = 6.9$. Interestingly, the mass M corresponding to the Lane-Emden function is *independent* of ρ_0, therefore *uniquely* determined by the proportionality factor K. More precisely, using (1.5) as well the equation (1.6), we can calculate

$$\begin{aligned} M &= 4\pi \int_0^R \rho(r) r^2 dr \\ &= 4\pi \left(\frac{\pi}{G}K\right)^{-3/2} \int_0^{x_r} x^2\Theta(x)^3 dx = 4\pi \left(\frac{\pi G}{K}\right)^{-3/2} \int_0^{x_r} \frac{d}{dx}(-x^2\Theta'(x))dx \\ &= 4\pi \left(\frac{\pi G}{K}\right)^{-3/2} x_r^2 |\Theta'(x_r)| \approx 1.4 M_\odot, \end{aligned} \tag{1.7}$$

where $|\Theta'(x_r)|$ can be computed numerically, see [**16**]. This *unique* mass M was denoted as critical, respectively limiting mass, M_C, by Chandrasekhar. Chandrasekhar further concluded that for all $M > M_C$ the stars are necessarily thermodynamically unstable, whereas for all smaller masses the stars are stable. A simplified instability argument is given in the following section.

1.1. Instability due to energy estimates. Actually, the occurrence of a gravitational collapse for relativistic systems interacting by Newtonian gravitation, can be obtained by a far simpler argument, with a different critical mass, but, still, reflecting the main idea of Chandrasekhar's original argument. Assume N electrons are occupying a volume V, with radius R, such that according to the Pauli-principle we attribute a volume V/N to each single electron. Heisenberg's uncertainty principle tells us that the mean electron momentum is

$$p \sim (V/N)^{-1/3} = 1/(R/N^{1/3}).$$

Remark, that we use units such that $c = \hbar = 1$. The energy momentum relation for a relativistic particle, according to Einstein, is given by

$$\varepsilon(p) = \sqrt{m_e^2 + p^2}.$$

Assuming further that the momenta of the particles in the star are much bigger than the electron rest mass m_e, such that $\varepsilon(p) \approx |p|$, then the total kinetic energy can simply be written as

$$T = T(R) = N\varepsilon(p) \approx N|p| \sim N^{4/3}/R.$$

Let us mention that in our calculation we assume that the kinetic energy is totally made up by the electrons in the star, meaning the movement of the nuclei are omitted which is justified by the fact that their momenta are negligible compared to the momenta of the electron. However, the gravitational energy is mainly dominated by the nucleons, such that the total mass of the system is roughly

$$M \approx 2Nm_n, \tag{1.8}$$

with m_n being the nucleon mass, the 2 standing for the fact that we assume to deal with systems having 2 nucleons per electron, such as stars built of Helium or carbon. Up to a factor of order 1 the gravitational energy is given by

$$V = V(R) = -G\frac{M^2}{R},$$

which is what you get if you calculate the gravitational interaction energy of two balls of radius R with constant density. Hence the total energy looks like

$$E(R) = T(R) + V(R) \approx N^{4/3}/R - GM^2/R.$$

In order the star to settle in an equilibrium state the energy has to attain its minimum value, which is only possible if

$$N^{4/3} \geqslant GM^2, \text{ or } \left(\frac{M}{2m_n}\right)^{4/3} \geqslant GM^2,$$

which gives the critical mass

$$\frac{1}{(2m_n)^2 G^{3/2}} \approx 1.8 M_\odot,$$

being only slightly off Chandrasekhar's, more precise, calculation.

1.2. Remark on neutron stars and the Tollmann-Oppenheimer-Volkov-equation. For neutron stars the mass density becomes so big that one has to take into account effects from general relativity. Solving the Einstein equations for the radially symmetric interior Schwarzschild solution, where the energy-momentum tensor depends on the pressure P and the density ρ of the system, Tollmann, Oppenheimer and Volkov derived the hydro-gravitational equilibrium equation, see, e.g. [**14, 16**]

$$\dot{P}(r) = -\frac{GM(r)\rho(r)}{r^2}\left[1 + \frac{P(r)}{\rho(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{M(r)}\right]\left[1 - \frac{2GM(r)}{r}\right]^{-1}. \tag{1.9}$$

Remark that in the limiting case of $P/\rho \ll 1$, which is basically equivalent to $r^3 P/M(r) \ll 1$, and $GM/r \ll 1$ all the brackets in (1.9) fall out and one obtains the classical Newtonian equilibrium equation (1.2).

Let us give some heuristic argument why these conditions are satisfied in the case of a white dwarf. For simplicity let us consider the boundary of the star at $r = R$. The corresponding mass $M(R) = M$ is then the total mass. The total pressure is basically given by $P = E_{\text{mat}}/R^3$, the pressure coming from the kinetic energy of the particles, which is roughly Nm_e/R^3 (using an individual energy of $m_e c^2$, $c = 1$ in our units). The total density is given by the mass of the nucleons, $\rho = 2Nm_n/R^3$, hence

$$P/\rho \sim m_e/m_n \sim 1/2000 \ll 1.$$

Further $2GM/R = r_S/R$, where r_S denotes the Schwarzschild radius[1] $r_S = 2GM$ of the mass M. For white dwarfs of the mass $M_\odot$ of the sun the ratio $r_S/R \sim 3 \cdot 10^{-4} \ll 1$, which shows that the Tolmann-Oppenheimer-Volkov equation reduces indeed to the classical equation (1.2). Let us mention that for stars with much higher density, like super massive neutron stars, the ration r_S/R is of the order 1, such that it becomes necessary to consider the effects coming from general relativity which is reflected in (1.9).

Let us finally remark that the equation (1.9) can be solved explicitly under the simple assumption that the density of the star is homogenous, i.e.

$$\rho(r) = \begin{cases} \rho_0 & (r \leqslant R) \\ 0 & (r > R) \end{cases}$$

Using the boundary condition that $P(r) = 0$ for $r \geqslant R$ the function $P(r)$ is given explicitly, see [**16**, (11.6.4)], depending on the parameters $M = M(R)$, the total mass, and R, the radius of the star. The function $P(r)$ is monotonically decreasing in r and the central pressure has the form

$$P(0) = \frac{3M}{4\pi R^3} \frac{[1 - (2MG/R)]^{1/2} - 1}{1 - 3\,[1 - (2MG/R)]^{1/2}}. \tag{1.10}$$

Being in the state of equilibrium requires that the central pressure attains a finite value, such that (1.10) implies the following equilibrium condition

$$\frac{MG}{R} < \frac{4}{9},$$

which is equivalent to $R > R_S\frac{9}{8}$. Let us emphasize here that according to general relativity one has a stability condition for stars, even in the case one assumes a homogeneous density. This is not true for Newtonian gravity, where homogenous stars are always perfectly stable. Hence, the effects of general relativity enhance instability [**16, 14, 3**].

2. Relativistic Hamiltonian for gravitating particles

In 1984 Lieb and Thirring [**13**] made an effort to derive Chandrasekhar's stability equation as well as the critical mass from first principles, starting from a quantum mechanical Hamiltonian. So they wrote down the Hamiltonian

$$H_{\kappa,N} = \sum_{i=1}^{N} \sqrt{-\Delta_i + m_e^2} - \kappa \sum_{1 \leqslant i < j \leqslant N} |x_i - x_j|^{-1}, \tag{2.1}$$

acting on $\wedge_{i=1}^N L^2(\mathbb{R}) \otimes \mathbb{C}^2$, with $\kappa = G(m_2 + 2m_n)^2$. In other words, Lieb Thirring thought of a white dwarf as a massive stellar object attaining the ground-state of a Hamiltonian, where the kinetic energy is given by the electrons and the Newtonian interaction is dominated by the nucleons. Observe that the Hamiltonian is expressed only in terms of the electrons, whereas the interaction energy is mainly given by the nucleons, however, expressed in terms of the electron-variables, which is possible due to the assumption of *local charge* neutrality. $H_{\kappa,N}$ would perfectly suite for neutron stars, except for the fact that effects from general relativity are

[1] Recall that the Schwarzschild radius r_S gives the radius below which the object is supposed to appear as a black hole.

discarded. Let us remark, that Fowler speculated already in 1927, shortly after Schrödinger presented his quantum mechanical equation, that the white-dwarf material might be best likened to a single gigantic molecule in the lowest quantum state. Denote

$$E^Q_{\kappa,N} = \inf_{\|\psi\|=1} (\psi, H_N \psi),$$

as the ground state energy of the Hamiltonian (2.1). Then there exists a critical number of particles $N_Q(\kappa)$, such that the Hamiltonian is unbounded from below for

$$N > N_Q(\kappa),$$

associated with a critical mass

$$M_Q(\kappa) = N_Q(\kappa)(2m_n + m_e).$$

Lieb and Thirring proved the Chandrasekhar value of a critical mass, with the correct exponent $G^{-3/2}$, up to a factor 4. A few years later Lieb and Yau [**11**] proved the exact critical value of Chandrasekhar in the sense that

$$M_Q(\kappa)/M_C \to 1, \quad \text{as} \quad G \to 0,$$

with $M_C \approx 0.77(m_n)^{-2} G^{-3/2}$ being the Chandraskhar critical value. Among other things they also show that, using (1.4), for a fixed mass $M \leqslant M_c$ there is only one solution for the equation (1.2). And for $M > M_c$ there is no solution. Their proof is based on the fact that the minimizer of the semiclassical Thomas-Fermi functional $\mathcal{E}^C_\kappa(\rho)$, in (4.6), is a solution to the equation (1.2). More precisely, differentiating the associated Euler-Lagrange equation with respect to the radial coordinate leads to the gravitational hydrostatic equilibrium equation (1.2). We refer to [**12**] for a nice review of the results obtained by Lieb and Yau. See also [**14**].

If one denotes further $E^C_\kappa(N)$ as the lowest energy of the semiclassical functional $\mathcal{E}^C_\kappa(\rho)$, under the restriction of total number N, then Lieb and Yau also showed that in the large N limit the semiclassical energy is the same as the full quantum mechanical energy, i.e. $E^C_\kappa(N)/E^Q_\kappa(N) \to_{N\to\infty} 1$, if one keeps $N\kappa^{3/2}$ fixed. A similar statement is shown to hold between the quantum mechanical ground state and the solution to the semiclassical equation (1.2). The statement about the energies tell us that for large N the behavior of the system can be well captured by a mean-field theory. This idea is used for the study of the dynamical collapse which is the main part of the present review.

3. Dynamical description of the stellar collapse in the Hartree-Fock approximation

The study of the dynamical collapse of stellar objects was initiated by Fröhlich and Lenzmann [**4, 5**]. To simplify matters, Fröhlich and Lenzmann considered the mean-field equation corresponding to the Hamiltonian $H_{\kappa,N}$, for the natural reason that in the large N limit the correlations are supposed to be of minor influence. Let us therefore first reduce $H_{\kappa,N}$ to *one body density matrices* γ, with $\mathrm{Tr}\gamma = N$ acting on $L^2(\mathbb{R}^3;\mathbb{C})$[2] The Fermi-character is expressed by

$$0 \leqslant \gamma \leqslant 1,$$

[2]For simplicity, we discard the spin of the particles throughout. But all our arguments can be easily generalized to particles having q internal degrees of freedom, where $L^2(\mathbb{R}^3;\mathbb{C})$ has to be replaced by $L^2(\mathbb{R}^3;\mathbb{C}^q)$.

as operators on $L^2(\mathbb{R}^3;\mathbb{C})$.

A specific class of γ's is given by density matrices stemming from slater determinants

$$\psi_N = \psi_1 \wedge \psi_2 \cdots \wedge \psi_N,$$

with

$$\gamma_{\psi_N}(x,y) = \sum_{i=1}^{N} \psi_i(x)\overline{\psi_i(y)}.$$

The energy expectation value for these states is

$$(\psi_N, H_{\kappa,N}\psi_N) = \mathcal{E}_{\mathrm{HF}}(\gamma_{\psi_N}),$$

where $\mathcal{E}_{\mathrm{HF}}(\gamma)$ denotes the Hartree-Fock energy functional which is given by

$$\mathcal{E}_{\mathrm{HF}}(\gamma) = \mathrm{Tr}(K\gamma) - \frac{\kappa}{2}\iint \frac{1}{|x-y|}\Big(\rho_\gamma(x)\rho_\gamma(y) - |\gamma(x,y)|^2\Big)\,dx\,dy\,. \tag{3.1}$$

Here $\rho_\gamma(x) = \gamma(x,x)$ is the density associated with the one-body density matrix γ. Since γ is assumed to be trace-class the density is well defined. Further the pseudo-differential operator

$$K = \sqrt{-\Delta + m_e^2}$$

describes the kinetic energy of a relativistic quantum particle with rest mass $m_e \geqslant 0$. The term $-\frac{\kappa}{2}\iint \frac{\rho_\gamma(x)\rho_\gamma(y)}{|x-y|}dxdy$ denotes the classical interaction energy, whereas $\frac{\kappa}{2}\iint \frac{|\gamma(x,y)|^2}{|x-y|}\,dx\,dy$ is known as the exchange term and a particular feature of the Pauli-principle.

We are primarily interested in the corresponding time-dependent HF equations, which can be formulated as the following initial-value problem:

$$\boxed{\begin{cases} i\dfrac{d}{dt}\gamma_t = [H_{\gamma_t}, \gamma_t]\,, \\ \gamma_{|t=0} = \gamma_0 \in \mathcal{K}_{\mathrm{HF}}. \end{cases}} \tag{3.2}$$

Here $[A,B] = AB - BA$ denotes the commutator and

$$H_\gamma := \sqrt{-\Delta + m^2} - \kappa\left(\frac{1}{|x|} * \rho_\gamma\right) + \kappa\frac{1}{|x-y|}\gamma(x,y) \tag{3.3}$$

is the so-called *mean-field operator* which depends on γ and acts on the one-body space $L^2(\mathbb{R}^3;\mathbb{C})$. Here and henceforth, the symbol $*$ stands for convolution of functions on $\mathbb{R}^3$. For expositional convenience, we use a slight abuse of notation by writing $\frac{1}{|x-y|}\gamma(x,y)$ for the operator whose integral kernel is the function $(x,y) \mapsto \frac{1}{|x-y|}\gamma(x,y)$. The appropriate set $\mathcal{K}_{\mathrm{HF}}$ of initial data for the evolution equation (3.2) will be defined below. We remark that the number of particles $\mathrm{Tr}(\gamma_t)$ and the energy $\mathcal{E}_{\mathrm{HF}}(\gamma_t)$ are both conserved along the flow given by (3.2).

Formulated differently, the evolution equation has actually been derived in a way such that the energy is conserved. In fact, in a very formal sense, we can write

$$\frac{d}{dt}\mathcal{E}_{\mathrm{HF}}(\gamma_t) = \mathrm{Tr}H_{\gamma_t}\dot{\gamma}_t = -i\mathrm{Tr}H_{\gamma_t}[H_{\gamma_t},\gamma_t] = -i\mathrm{Tr}[H^2_{\gamma_t},\gamma_t] = 0,$$

since the trace of a commutator vanishes. A similar argument shows that $\mathrm{Tr}\gamma_t$ is a conserved quantity.

Let us now define the evolution equation in a precise form. To this end, we denote by $\mathfrak{S}_p$ the space of operators A acting on $L^2(\mathbb{R}^3,\mathbb{C})$ such that $\|A\|^p_{\mathfrak{S}_p} = \mathrm{Tr}|A|^p < \infty$. Further we introduce a set of fermionic density matrices

$$\mathcal{K}_{\mathrm{HF}} := \{\gamma = \gamma^* \in \mathcal{X}_{\mathrm{HF}} : 0 \leqslant \gamma \leqslant 1\}\,, \tag{3.4}$$

where the Sobolev-type space $\mathcal{X}_{\mathrm{HF}}$ is defined by

$$\mathcal{X}_{\mathrm{HF}} := \{\gamma \in \mathfrak{S}_1 : \|\gamma\|_{\mathcal{X}_{\mathrm{HF}}} < \infty\} \tag{3.5}$$

with the norm

$$\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}} := \left\|(m_e^2-\Delta)^{1/4}\gamma(m_e^2-\Delta)^{1/4}\right\|_{\mathfrak{S}_1}\,. \tag{3.6}$$

The norm simply corresponds to the kinetic energy $\mathrm{Tr}\sqrt{-\Delta+m_e^2}\gamma$. By means of standard PDE methods it is straight forward to show that the initial-value problem (3.2) is locally well-posed in $\mathcal{K}_{\mathrm{HF}}$.

Theorem 3.1 (Well-posedness in Hartree-Fock theory). *For each initial datum $\gamma_0 \in \mathcal{K}_{\mathrm{HF}}$, there exists a unique solution $\gamma_t \in C^0([0,T),\mathcal{K}_{\mathrm{HF}}) \cap C^1([0,T);\mathcal{X}'_{\mathrm{HF}})$ with maximal time of existence $0 < T \leqslant \infty$. Finally, if the maximal time satisfies $T < \infty$, then*

$$\mathrm{Tr}(-\Delta+m_e^2)^{1/2}\gamma_t \to \infty$$

as $t \uparrow T$.

This theorem was proved in [**5**]. The theorem tells us that there is always a local in time solution provided we start with smooth enough initial data, and it also says that if the solution cannot be extended for all times, then it has to blow up in the $\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}}$-norm within finite time.

According to Chandrasekhar a white dwarf is in a stable equilibrium state if its mass is below the critical mass M_c. This means that if the number of particles N is smaller than $N_c = M_c/(2m_n+m_e)$ we expect the evolution equation to converge to an equilibrium state, or at least to stay bounded in the $\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}}$-norm.

Such a statement can be proven, [**5**, Theorem 2], for $N < N_c^* < N_c$, with N_c^* being a critical value which pops out of the proof. Let us indicate why we do not have the proof up to the critical value N_c. As usually in the PDE business, one needs an a-priori bound for $\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}}$ to guarantee that it cannot explode. This a-priori bound usually comes from a conserved quantity such as the energy. In other words one has to make sure that

$$0 < \mathcal{E}_{\mathrm{HF}}(\gamma_0) = \mathcal{E}_{\mathrm{HF}}(\gamma_t) \geqslant \delta\|\gamma_t\|_{\mathcal{X}_{\mathrm{HF}}}, \tag{3.7}$$

for some $\delta > 0$, which then lifts the local solution to a (bounded) global one. The energy is known to be non-negative as long as $N < N_c$. To be able to bound the energy by the norm, one needs a bound on $D(\rho_\gamma,\rho_\gamma)$ in terms of the kinetic energy. This can be achieved via the Hardy-Littlewood-Sobolev-inequality [**9**], together with the relativistic Lieb-Thirring inequality. However the known constant $\tilde{K}$ in the bound for the corresponding Lieb-Thirring inequality, see [**2, 10**],

$$\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}} \geqslant \tilde{K}\int_{\mathbb{R}^3}\rho_\gamma^{4/3}, \tag{3.8}$$

is slightly worse than the semiclassical constant K in (4.5), which is the actual reason for not having a proof up to N_c. Combining the inequality

$$D(\rho,\rho) \leqslant \bar{c}N^{2/3}\int_{\mathbb{R}^3}\rho^{4/3}dx,$$

with sharp constant $\bar{c} \approx 1.092$, see [**11**], with the Lieb-Thirring bound (3.8), we get

$$\mathcal{E}_{\mathrm{HF}}(\gamma_t) \geqslant (1-\kappa\bar{c}N^{2/3}\tilde{K})\|\gamma_t\|_{\mathcal{X}_{\mathrm{HF}}},$$

which satisfies (3.7), for $N < N_c^* = (\kappa\bar{c}\tilde{K})^{-3/2}$.

This can be summarized as follows [**5**, Theorem 1]

Theorem 3.2 (Global in time solutions). *Every solution of Theorem 3.1 with initial condition* $\mathrm{Tr}\gamma_0 < N_c^*$ *exists for all times* $0 \leqslant t < \infty$ *and the norm* $\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}}$ *is uniformly bounded.*

A particular class of global-in-time solutions of (3.2) are *stationary states* satisfying $[H_\gamma,\gamma]=0$. Important examples for such stationary states are given by the minimizers of $\mathcal{E}_{\mathrm{HF}}(\gamma)$, subject to the constraint $\mathrm{Tr}(\gamma)=N$ with N not too large, which were proven to exist in [**8**].

Lets now turn to the more interesting question of the situation where the mass of the initial state is bigger than M_C, respectively the number of particles N larger than N_c. Then the energy functional $\mathcal{E}_{\mathrm{HF}}(\gamma)$ is no longer bounded from below, and we can find states which have a negative energy. For such initial data, together with the requirement that the initial state γ is radial, one can show collapse. More precisely one can show that the solution γ_t of (3.2) blows up in the $\|\cdot\|_{\mathcal{X}_{\mathrm{HF}}}$-norm after finite time.

Let us first make precise what we understand under spherically symmetric, respectively radial, states. We say that $\gamma \in \mathcal{K}_{\mathrm{HF}}$ is *spherically symmetric* when

$$\gamma(Rx,Ry)=\gamma(x,y) \quad \text{for all} \quad x,y\in\mathbb{R}^3 \quad \text{and all } R\in SO(3),$$

where $SO(3)$ denotes the set of all rotations.

It is not difficult to verify that spherical symmetry of γ_t is preserved under the flow (3.2). We also note that, if γ_0 is sufficiently regular, then the condition of spherical symmetry can also be written as the commutator condition

$$[\gamma_t, L]=0. \tag{3.9}$$

Here $L=-ix\wedge\nabla_x$ is the angular momentum operator, and $\wedge$ denotes the cross product on $\mathbb{R}^3$. This holds because any rotation $R_{n,t}$ around the unit vector n about an angle δ can be written in the form of

$$(e^{i\delta n\cdot L}\gamma e^{-i\delta n\cdot L})(x,y)=\gamma(R_{n,\delta}x,R_{n,\delta}y)=\gamma(x,y),$$

where in the last equality we assumed γ to be radial. Differentiating this relation with respect to δ leads to (3.9).

Our main statement in the following concerns the blow-up question.

Theorem 3.3 (Blowup in Hartree-Fock theory). *Let* $\gamma_0\in\mathcal{K}_{\mathrm{HF}}$ *be spherically symmetric and suppose that*

$$\mathrm{Tr}\,|x|^4\gamma_0+\mathrm{Tr}\,(-\Delta)\gamma_0+\mathrm{Tr}\,|L|^2\gamma_0<\infty$$

where $L=-ix\wedge\nabla_x$ *denotes the angular momentum operator. Our conclusion is the following: If* γ_0 *has negative energy, that is* $\mathcal{E}_{\mathrm{HF}}(\gamma_0)<0$*, then the corresponding* *solution* γ_t *to (3.2) blows up in finite time; i. e., we have* $\mathrm{Tr}\,(-\Delta+m_e^2)^{1/2}\,\gamma_t \to \infty$ *as* $t\uparrow T$ *for some* $T<\infty$.

The regularity condition should simply assure that the quantities we are looking at, such as $\mathrm{Tr}M\gamma_t$ and $\mathrm{Tr}|L|^2\gamma_t$, remain well defined as long as γ_t exists. Such a blow-up result for the Hartree-equation, meaning neglecting the exchange term in (3.2), under the further assumption that the initial state γ_0 is an orthogonal projection of rank N, was first proven by Fröhlich and Lenzmann [**4, 5**]. Hainzl and Schlein [**7**] extended the blow-up result to the Hartree-Fock equation, by using an additional conservation law for the square of the angular momentum $\mathrm{Tr}(|L|^2\gamma_t)$. However the assumption of finite rank was still important in the work of [**7**]. Among other things this condition was removed by Hainzl, Lenzmann, Lewin, Schlein in [**6**], where the proof of the the blow up result as stated in Theorem 3.3 is presented.

This shows the onset of a dynamical collapse of a white dwarf in the case that the initial mass is too large. The fact, that this blow-up result actually implies a mass concentration at the origin at the time of collapse, was shown by Fröhlich and Lenzmann in [**5**, Theorem 4] and presented in the following statement.

Theorem 3.4 (Mass concentration for radial blow-up). *Let* γ_t *be a radial solution of* (3.2)*, that blows up at finite time* $T>0$*. Then, for any* $R>0$,

$$\liminf_{t\uparrow T}\int_{|x|\leqslant R}\rho_{\gamma_t}(x)dx \geqslant N_c^*,$$

with N_c^* *given above.*

In the following we want to sketch the proof of Theorem 3.3.

Sketch of the proof of Theorem 3.3. The main strategy of the proof is virial-type argument. Since the relativistic virial operator [**4, 5**]

$$M = x\cdot\sqrt{-\Delta+m_e^2}x = \sum_{i=1}^{3} x_i\sqrt{-\Delta+m_e^2}x_i$$

is *non-negative*, it suffices to show that the following bound holds for radially symmetric solutions γ_t,

$$0\leqslant \mathrm{Tr}M\gamma_t \leqslant 2t^2\mathcal{E}_{\mathrm{HF}}(\gamma_0) + CNt\,\mathrm{Tr}(1+|L|^2)\gamma_0 + C, \tag{3.10}$$

with some constant C. This necessarily yields a maximal time T of existence if the initial energy is negative $\mathcal{E}_{\mathrm{HF}}(\gamma_0)<0$. In other words the norm $\|\gamma\|_{\mathcal{X}_{\mathrm{HF}}}$ blows up after time T. The way to obtain (3.10) is essentially by differentiating $\mathrm{Tr}M\gamma_t$ twice. More precisely, we have

$$\frac{d}{dt}\mathrm{Tr}M\gamma_t = i\mathrm{Tr}[H_{\gamma_t},M]\gamma_t.$$

To this aim let us first look at the first term in this expression which involves the commutator

$$\begin{aligned}(3.11)\quad &\sum_{j=1}^{3} i[\sqrt{p^2+m_e^2}, x_j\sqrt{p^2+m_e^2}x_j] \\ &= \sum_{j=1}^{3} i\left([\sqrt{p^2+m_e^2}, x_j]\sqrt{p^2+m_e^2}x_j - x_j\sqrt{p^2+m_e^2}[x_j,\sqrt{p^2+m_e^2}]\right) \\ &= x\cdot p + p\cdot x = 2A.\end{aligned}$$

Here we used the representation $x_j = i\partial_{p_j}$, and the consequent commutator relation

$$[x_j, \sqrt{p^2+m_e^2}] = ip/\sqrt{p^2+m_e^2}.$$

The operator

$$A = \frac{1}{2}(x\cdot p + p\cdot x) = \frac{3}{2i} + x\cdot p,$$

is called the dilation operator, for the simple reason that it is the generator of the group of dilations, in the sense that, for $x\in\mathbb{R}^3$,

$$(e^{i\theta A}\psi)(x) = e^{3\theta/2}\psi(e^\theta x).$$

Differentiating the last equation confirms the formula for A. Hence, one obtains

$$(3.12)\qquad \frac{d}{dt}\mathrm{Tr}M\gamma_t = 2\mathrm{Tr}A\gamma_t - i\mathrm{Tr}\left[V*\rho_{\gamma_t}, M\right]\gamma_t + i\mathrm{Tr}\left[R_{\gamma_t}, M\right]\gamma_t,$$

where we used $V(x) = \frac{\kappa}{|x|}$, and

$$R_\gamma(x,y) = V(|x-y|)\gamma(x,y).$$

The hardest part of the proof is to show, for radial γ, that

$$(3.13)\quad \left|\mathrm{Tr}\left[M,\left(V*\rho_\gamma\right)\right]\gamma\right| + \left|\mathrm{Tr}\left[M,R_\gamma\right]\gamma\right| \leqslant C\left(\mathrm{Tr}(\gamma)\right)^2 + C\,\mathrm{Tr}(\gamma)\,\mathrm{Tr}(1+|L|^2)\gamma,$$

for which we refer the interested reader to [**6**], where we essentially use the fact that radially symmetric functions can be expressed in terms of Legendre polynomials, which explicitly reflect the angular momentum dependence.

Equation (3.13) implies that

$$(3.14)\qquad \frac{d}{dt}\,\mathrm{Tr}\,M\gamma_t \leqslant 2\mathrm{Tr}\,A\gamma_t + C\mathrm{Tr}(\gamma_0)\,\mathrm{Tr}(1+|L|^2)\gamma_0$$

For deriving the desired inequality (3.10) it suffices to show

$$(3.15)\qquad \frac{d}{dt}\mathrm{Tr}A\gamma_t \leqslant \mathcal{E}_{\mathrm{HF}}(\gamma_t) = \mathcal{E}_{\mathrm{HF}}(\gamma_0),$$

which even holds with equality, in the case $m_e = 0$, which we want to indicate in the following. If $m_e = 0$ one has

$$\begin{aligned}(3.16)\quad &\frac{d}{dt}\mathrm{Tr}A\gamma_t = i\mathrm{Tr}[H_{\gamma_t}, A]\gamma_t = \frac{d}{d\theta}\mathrm{Tr}H_{\gamma_t}e^{i\theta A}\gamma_t e^{-i\theta A}\Big|_{\theta=0} = \frac{d}{d\theta}\mathcal{E}_{\mathrm{HF}}\left(e^{i\theta A}\gamma_t e^{-i\theta A}\right)\Big|_{\theta=0} \\ &= \frac{d}{d\theta}e^\theta\mathcal{E}_{\mathrm{HF}}(\gamma_t)\Big|_{\theta=0} = \mathcal{E}_{\mathrm{HF}}(\gamma_t),\end{aligned}$$

where used the scaling properties of the energy functional $\mathcal{E}_{\mathrm{HF}}$ evaluated for states γ with the kernel

$$\left(e^{i\theta A}\gamma_t e^{-i\theta A}\right)(x,y) = e^{3\theta}\gamma_t(e^\theta x, e^\theta y).$$

□

4. Appendix

4.1. Equation of state for a relativistic Fermi gas at 0 temperature. Consider a degenerate Fermi-gas at the temperature $T = 0$. Fix a volume V, cube or ball, and fill it with electrons having relativistic energy. Since we are at the ground state, according to the Pauli principle each (Dirichlet, or Neumann)-eigenstate will be occupied twice (due to the spin 1/2.) Let us fill the system up to the *Fermi-momentum* p_F. To a good approximation the energy as well as the number of particles can be calculated semiclassically, namely using the phase-space approach. The number of particles lying under the Fermi-momentum are accordingly given by

$$N = \frac{1}{(2\pi)^3} \int_V d^3x \int_{|p| \leqslant p_F} 2\, d^3p \;=\; \frac{2V}{(2\pi)^3} \frac{4\pi}{3} p_F^3, \tag{4.1}$$

the 2 accounting for the Pauli principle, which relates the Fermi-momentum to the electron density $n_e = N/V$, via

$$p_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3} = (3\pi^2 n_e)^{1/3}. \tag{4.2}$$

The energy of the electron-gas, semiclassically, is given by

$$E_{\mathrm{mat}} \;=\; \frac{1}{(2\pi)^3} \int_V d^3x \int_{|p| \leqslant p_F} 2\varepsilon(p) d^3p \;=\; \frac{2V}{(2\pi)^3} 4\pi \int_0^{p_F} p^2 \sqrt{m_e^2 + p^2}\, dp. \tag{4.3}$$

Remember now that according to thermodynamics the pressure satisfies the relation

$$dE = TdS - PdV.$$

For $T = 0$ and $E = E_{\mathrm{mat}}$, we therefore have, see also [**15**, 2.3.18],

$$\begin{aligned} P_{\mathrm{mat}} &= -\frac{\partial E_{\mathrm{mat}}}{\partial V} \\ &= -\frac{2}{(2\pi)^3} 4\pi \int_0^{p_F} p^2 \sqrt{m_e^2 + p^2}\, dp - \frac{2V}{(2\pi)^3} 4\pi p_F^2 \sqrt{m_e^2 + p_F^2}\, \frac{\partial p_F}{\partial V} \\ &= -\frac{1}{\pi^2} \int_0^{p_F} p^2 \sqrt{m_e^2 + p^2}\, dp + \frac{1}{\pi^2} \frac{p_F^3}{3} \sqrt{m_e^2 + p_F^2}\, \frac{\partial p_F}{\partial V} \\ &= \frac{1}{3\pi^2} \int_0^{p_F} \frac{p^4}{\sqrt{m_e^2 + p^2}}\, dp, \end{aligned} \tag{4.4}$$

where we used that

$$\frac{\partial p_F}{\partial V} = -(3\pi^2)^{1/3} \frac{1}{3} \frac{N^{1/3}}{V^{4/3}} = -(3\pi^2)^{1/3} \frac{1}{3} \frac{n_e^{1/3}}{V} = -\frac{1}{3} \frac{p_F}{V},$$

and we used the elementary fact that $g(p_F) = \int_0^{p_F} g'(p) dp$, if $g(0) = 0$. In the extreme relativistic case, Chandrasekhar was interested in, i.e. $\sqrt{m_e^2 + p^2} \sim |p|$, the pressure P_{mat} in (4.4) becomes

$$P_{\mathrm{mat}} \simeq \frac{1}{12\pi^2} p_F^4 = \frac{1}{12\pi^2} (3\pi^2)^{4/3}\, n_e^{4/3}.$$

This is the kinetic pressure in terms of the charge density. However, we want to have this in terms of the *mass density* ρ,

$$\rho = \frac{M}{V} \simeq \frac{2Nm_n}{V} = 2n_e m_n,$$

which is given by

$$\boxed{P_{\text{mat}} \simeq \frac{1}{12\pi^2}\left(\frac{3\pi^2}{2m_n}\right)^{4/3}\rho^{4/3}}. \tag{4.5}$$

4.2. The semiclassical energy functional. We now also define the semiclassical energy functional $\mathcal{E}_\kappa^C(\rho)$, where C stands for Chandrasekhar. This time we write the functional in term of the *charge density* ρ, we use however the same symbol as for the mass density in the previous chapter. The kinetic energy of a free Fermi-gas, given in terms of the charge density

$$\rho = n_e = N/V$$

can be read of from (4.3), i.e.,

$$E_{\text{mat}} = \frac{V}{\pi^2}\int_0^{\left(3\pi^2\rho\right)^{1/3}} p^2\sqrt{m_e^2+p^2}\,dp.$$

Allowing the density $\rho = \rho(x)$ to vary in space this corresponds to the kinetic energy $\int_{\mathbb{R}^3} j(\rho(x))d^3x$, with

$$j(\rho(x)) := \frac{1}{\pi^2}\int_0^{\left(3\pi^2\rho\right)^{1/3}} p^2\sqrt{m_e^2+p^2}\,dp.$$

To write the interaction energy in terms of the charge density, remember that we had

$$V(R) \approx -G\frac{M^2}{R} \approx -G(2m_n)^2\frac{N^2}{R} = -\kappa\frac{N^2}{R},$$

which corresponds in the case of non-constant densities ρ to the gravitational energy

$$D(\rho,\rho) = \frac{\kappa}{2}\iint \frac{\rho(x)\rho(y)}{|x-y|}dxdy,$$

such that the total energy-functional, known as relativistic Thomas Fermi-functional, is of the form

$$\mathcal{E}_\kappa^C(\rho) = \int_{\mathbb{R}^3} j(\rho(x))d^3x - D(\rho,\rho), \tag{4.6}$$

where $\kappa = G(2m_n+m_e)^2$.

References

1. S. Chandrasekhar. *Phil. Mag.* **11** 592 (1931); *Astrophys. J.* **74**, 81 (1931); *Rev. Mod. Physics* **56**, 137 (1984)
2. I. Daubechies. An uncertainty principle for fermions with generalized kinetic energy. *Commun. Math. Phys.* **90** 511–520, (1983)
3. T. Fliessbach. Allgemeine Relativitätstheorie. *Spektrum Akademischer Verlag* Heidelberg (2006).
4. J. Fröhlich, E. Lenzmann. Blowup for Nonlinear Wave Equations describing Boson Stars. *Comm. Pure Appl. Math.* **60** (2007), no. 11, 1691–1705.
5. J. Fröhlich, E. Lenzmann. Dynamical Collapse of White Dwarfs in Hartree- and Hartree-Fock Theory. *Comm. Math. Phys.* 274 (2007), no. 3, 737-750.

6. C. Hainzl, E. Lenzmann, M. Lewin, B. Schlein. On blowup for the time-dependent generalized Hartree-Fock equations. arXiv: 0909.3043
7. C. Hainzl, B. Schlein. Stellar Collapse in the time-dependent Hartree-Fock approximation. *Commun. Math. Phys.* **287**, 705–717 (2009).
8. E. Lenzmann, M. Lewin. Minimizers for the Hartree-Fock-Bogolubov theory of neutron stars and white dwarfs. To appear in *Duke Journal*, arXiv: 0809.2560 (2008)
9. E. H. Lieb, M. Loss. Analysis *Graduate studies in Mathematics* AMS, Providence, RI (2001)
10. E.H. Lieb, R. Seiringer, The Stability of Matter in Quantum Mechanics, *Cambridge Univ. Press* (2010)
11. E.H. Lieb, H. T. Yau. The Chandrasekhar Theory of stellar collapse as the limit of quantum mechanics. *Commun. Math. Phys.* **112**, 147-174 (1987)
12. E.H. Lieb, H. T. Yau. A rigorous examination of the Chandrasekhar theory stellar collapse. *Astrophys. Jour.* **323**, 140-144 (1987)
13. E. L. Lieb, W. Thirring. Gravitational Collapse in quantum mechanics with relativistic kinetic energy. *Ann. Phys.* **155**, 494-512 (1984)
14. N. Straumann. General relativity, *Texts and Monographs in Physics* Spriner, 2004.
15. W. Thirring. Atoms molecules and large systems. *Springer* second Edition. (2002)
16. S. Weinberg. Gravitation and Cosmology. *John Wiley and Sons* New York (1972).

Department of Mathematics, UAB, Birmingham, AL 35294-1170, USA.
E-mail address: hainzl@math.uab.edu